# DERIVATION OF A FEW SOLVABLE POTENTIALS FOR THE SCHRÖDINGER EQUATION

*by* I. A. KHAN, *Department of Mathematics, Government Engineering College, Rewa (M.P.)*

(Communicated by K. S. G. Doss, F.N.A.)

(*Received 23 September 1971; after revision 21 June 1972*)

Making use of the second order of generalized associated Legendre and Stokes's differential equation, a few solvable potentials for the $S$-wave Schrödinger equation, have been evaluated. Solutions for some of the potentials and their corresponding expressions for $S$-matrices have also been established. Some known potentials fall as the particular cases of one of the potentials given here.

## 1. INTRODUCTION

A procedure for constructing phase equivalent potentials for which Schrödinger equation may be solved in terms of elementary functions, was introduced by Bargmann (1949). Employing the suitable functional transformations to a second order differential equation, Bhattacharjie and Sudarshan (1962) formulated a general method for evaluating soluble potentials for the Schrodinger equation. Subsequently, Bose (1963) developed another general technique for deriving the potentials, by considering the normal form of the second order differential equation and using the Schwarzian derivative. In the latter two cases, the potentials were obtained from hypergeometric, confluent hypergeometric and Bassel differential equations.

Following the Bose's technique and using the modified Mathieu equation, Aly and Spector (1965) evaluated an inverse fourth power singular potential. Sharma and Varma (1970) derived some solvable potentials through Whittaker and Jacobi differential equations with the help of the procedure of Bhattacharjie and Sudarshan. Moreover, Sharma (1970*a*, *b*) obtained further solvable potentials from modified Mathieu and associated Legendre equations respectively, by using both the general techniques.

The object of this paper is to derive some periodic and exponential solvable potentials for $S$-save Schrodinger equation through generalized associated Legendre (1957) and Stokes's (1958) differential equations, following the method of Bhattacharjie and Sudarshan. The same potentials are also

evaluated using the Bose's technique, thus proving the fact that the two methods absolutely agree with each other.

Finally, the solutions for some of the potentials and their corresponding expressions for $S$-matrices have also been obtained.

One of our potentials (2.16) generalizes the potentials which were found by Sharma and Varma (1970) and Sharma (1970*b*) with the help of Jacobi, ultraspherical and associated Legendre equations.

## 2. Bhattacharjia and Sudarshan technique

(A). The functions $P_\lambda^{m,n}(z)$ and $Q_\lambda^{m,n}(z)$ two specified linearly independent solutions of the generalized associated Legendre differential equation

$$(1-z^2)\frac{d^2w(z)}{dz^2}-2z\frac{dw(z)}{dz}+\left\{\lambda(\lambda+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w(z)=0 \quad \ldots \quad (2.1)$$

have been introduced by Kuipers and Meulenbeld (1957) for complex values of the parameters $\lambda, m$ and $n$.

Following the relations [Braaksma and Meulenbeld 1967, (0.2) and (1.3)] for the values of $P_\lambda^{m,n}(z)$ and $Q_\lambda^{m,n}(z)$ and general solution of (2.1) can be written as

$$\begin{aligned}w(z) &= AP_\lambda^{m,n}(z)+BQ_\lambda^{m,n}(z)\\ &= A(z-1)^{\frac{-m}{2}}(z+1)^{\frac{n}{2}}\frac{1}{\Gamma(1-m)}\,{}_2F_1\left[\begin{matrix}1+\lambda-\frac{m}{2}+\frac{n}{2},\ -\lambda-\frac{m}{2}+\frac{n}{2}\,;\\ 1-m;\end{matrix}\ \frac{1-z}{2}\right]\\ &+Be^{\pi im}(-1)^{\frac{2\lambda+1}{2}}2^{+\frac{n}{2}-\frac{m}{2}}\frac{\Gamma\left(1+\lambda+\frac{m}{2}+\frac{n}{2}\right)\Gamma\left(1+\lambda+\frac{m}{2}-\frac{n}{2}\right)}{\Gamma(2+2\lambda)}\\ &\times(1+z)^{\frac{-2\lambda-2+m}{2}}(z-1)^{\frac{-m}{2}}{}_2F_1\left[\begin{matrix}\frac{2\lambda+n-m+2}{2},\ \frac{2\lambda-n-m+2}{2}\,;\\ 2\lambda+2;\end{matrix}\ \frac{2}{1+z}\right] \quad \ldots \quad (2.2)\end{aligned}$$

where $A$ and $B$ are arbitrary constants.

Equ. (2.1) can also be written as

$$\frac{d}{dz}[(1-z^2)w'(z)]+\left\{\lambda(\lambda+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w(z)=0 \quad \ldots \quad \ldots \quad (2\ 3)$$

where prime denotes the differentiation.

Employing the transformation $\rho = \frac{1}{2}\log\frac{1+z}{1-z}$ the equ. (2.3) assumes the following form

$$w''(\rho)+\left[\frac{4\lambda(\lambda+1)e^{2\rho}-m^2e^{2\rho}(e^{2\rho}+1)-n^2(e^{2\rho}+1)}{(e^{2\rho}+1)^2}\right] w\rho = 0 \quad \ldots \quad (2.4)$$

which is suitable for the purpose of constructing the soluble potentials.

By making the following substitutions

$$\rho = f(r),\ w(\rho) = g(r)\phi(r),\ h(r) = \frac{d}{dr}\{\log_e g(r)\} \quad \ldots \quad \ldots \quad (2.5)$$

the equ. (2.4) takes the form

$$\phi''(r)+A(r)\phi'(r)+B(r)\phi(r) = 0 \quad \ldots \quad \ldots \quad (2.6)$$

where

$$A(r) = \frac{2g'(r)}{g(r)} - \frac{f''(r)}{f'(r)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

$$B(r) = \frac{g''(r)}{g(r)} - \frac{g'(r)f''(r)}{g(r)f'(r)}$$

$$+\frac{4\lambda(\lambda+1)[f'(r)]^2e^{2f(r)}-(1+e^{2f(r)})(m^2e^{2f(r)}+n^2)[f'(r)]^2}{(e^{2f(r)}+1)^2}. \quad \ldots \quad (2.8)$$

Now, for equ. (2.6) to be the following $S$-wave radial Schrödinger equ.

$$\phi''(r)+[K^2-V(r)]\phi(r) = 0, \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

[adopting units such that $E = K^2$]

the values of $A(r)$ and $B(r)$ should be chosen such that

$$A(r) = 0, \quad B(r) = K^2-V(r) \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

where $K^2$ is purely parametric term.

Hence, from (2.7) and (2.10) we have

$$f'(r) = cg^2(r) \quad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

where $c$ is any arbitrary constant.

Similarly, with the help of (2.8) and (2.10) we have

$$h'(r)-h^2(r)+\frac{4\lambda(\lambda+1)[f'(r)]^2e^{2f(r)}-(e^{2f(r)}+1)(m^2e^{2f(r)}+n^2)[f'(r)]^2}{(e^{2f(r)}+1)^2} \equiv K^2-V(r). \quad \ldots \quad (2.12)$$

I. Considering as a particular choice

$$\rho = f(r) = \frac{1}{2} \log \frac{1+e^{-\alpha r/2}}{1-e^{-\alpha r/2}} \quad \ldots \quad \ldots \quad (2.13)$$

we obtain from (2.12)

$$\frac{\alpha^2}{2} \frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} - \frac{\alpha^2}{16} - \frac{\alpha^2}{4} \frac{e^{-\alpha r}}{(1-e^{-\alpha r})} - \frac{\alpha^2}{4} \frac{e^{-2\alpha r}}{(1-e^{-\alpha r})^2} + \frac{\lambda(\lambda+1)\alpha^2}{4} \frac{e^{-\alpha r}}{(1-e^{-\alpha r})}$$

$$- \frac{\alpha^2}{8} \frac{\{m^2+n^2+e^{-\alpha r/2}(m^2-n^2)\}e^{-\alpha r}}{(1-e^{-\alpha r})^2} \equiv K^2 - V(r) \quad \ldots \quad \ldots \quad \ldots \quad (2.14)$$

which suggests

$$K^2 = -\alpha^2/16 \quad \ldots \quad \ldots \quad \ldots \quad (2.15)$$

and

$$V(r) = -\left\{ \frac{A_1 e^{-\alpha r} + B_1 e^{-2\alpha r} + C_1 e^{-3\alpha r/2}}{(1-e^{-\alpha r})^2} \right\}. \quad \ldots \quad \ldots \quad (2.16)$$

where

$$A_1 = \frac{\alpha^2}{4} \left[ 1+\lambda(\lambda+1) - \frac{m^2+n^2}{2} \right], \; B_1 = \frac{-\lambda(\lambda+1)\alpha^2}{4}, \; C_1 = \frac{(n^2-m^2)\alpha^2}{8}. \quad (2.17)$$

Replacing $\alpha$ by $2\alpha$ and making the corresponding changes in (2.13) to (2.17), the potential derived in (2.16) assumes the following form

$$V(r) = -\left\{ \frac{A_2 e^{-2\alpha r} + C_2 e^{-3\alpha r} + B_2 e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} \right\} \quad \ldots \quad \ldots \quad (2.18)$$

where $A_2$, $B_2$ and $C_2$ are corresponding values of $A_1$, $B_1$ and $C_1$.

For $n = m$ the potential obtained in (2.16) yields exactly the same result due to Sharma (1970*b*, (24)), which is recgognized as Manning-Rosen-Newing potential energy function proposed for diatomic molecules (Manning and Rosen 1933; Newing 1935, 1940).

Further, replacing $e^{\alpha r}$ by $2e^{\alpha r}$ and making the correspondig changes in (2.13) to (2.17), the potential derived in (2.16) can be put in the following simplified form

$$V(r) = -\left\{ \frac{2A_1 e^{\alpha r} + B_1 + \sqrt{2} C_1 e^{\alpha r/2}}{(2e^{\alpha r}-1)^2} \right\}. \quad \ldots \quad \ldots \quad (2.19)$$

For $n = m$, the result (2.19) yields the potential of the same form which was given by Sharma and Varma [1970, (3.13) and (4.3)].

II. Thus, for the choice

$$\rho = f(r) = \frac{1}{2} \log \frac{1+\tan \alpha r}{1-\tan \alpha r} \quad \ldots \quad \ldots \quad \ldots \quad (2.20)$$

the potential, derived, with the help of (2.11) and (2.12) is

$$V(r) = -[A_3 \tan^2\alpha r \sec^2 2\alpha r + B_3 \tan^2 2\alpha r + C_3 \tan \alpha r \sec^2 2\alpha r] \quad \ldots \quad (2.21)$$

where

$$A_3 = -\lambda(\lambda+1)\alpha^2,\ B_3 = \frac{\alpha^2}{2}[2+2\lambda(\lambda+1)-m^2-n^2],\ C_3 = (n^2-m^2)\frac{\alpha^2}{2}$$

and

$$K^2 = \frac{\alpha^2}{2}[4+2\lambda(\lambda+1)-m^2-n^2]. \quad \ldots \quad \ldots \quad (2.22)$$

If we take $n = m$, the potential derived in (2.21) takes the form

$$V(r) = -[A_3 \tan^2 \alpha r \sec^2 2\alpha r + D \tan^2 2\alpha r] \quad \ldots \quad \ldots \quad (2.23)$$

where

$$D = \alpha^2(1+\lambda(\lambda+1)-m^2) \text{ and } K^2 = \alpha^2(2+\lambda(\lambda+1)-m^2). \quad \ldots \quad (2.24)$$

III. Similarly, for the choice

$$\rho = f(r) = \frac{1}{2}\log\frac{1+\sin\alpha r}{1-\sin\alpha r} \quad \ldots \quad \ldots \quad \ldots \quad (2.25)$$

and employing the eqns. (2.11) and (2.12), we construct the following potential

$$V(r) = -[A_4 \tan^2 \alpha r + B_4 \sin \alpha r \sec^2 \alpha r] \quad \ldots \quad \ldots \quad (2.26)$$

where

$$A_4 = \frac{\alpha^2}{4} - \frac{\alpha^2}{2}(m^2+n^2),\ B_4 = \frac{\alpha^2(n^2-m^2)}{2},\ K^2 = \frac{\alpha^2}{2}[1+2\lambda(\lambda+1)-m^2-n^2)]. \quad \ldots \quad (2.27)$$

Choosing $n = m$, the potential constructed in (2.26), assumes the form

$$V(r) = -D_1 \tan^2 \alpha r \quad \ldots \quad \ldots \quad (2.28)$$

where

$$D_1 = \frac{\alpha^2}{4} - \alpha^2 m^2 \text{ and } K^2 = \frac{\alpha^2}{2}[1+2\lambda(\lambda+1)-2m^2]. \quad \ldots \quad (2.29)$$

If we replace $\alpha r$ by $\pi/2-\alpha r$ in the transformation (2.25) and making the corresponding changes wheresoever required, the potentials derived in (2.26) and (2.28) are changed to the form

$$V(r) = -[A_4 \cot^2 \alpha r + B_4 \cos \alpha r \operatorname{cosec}^2 \alpha r] \quad \ldots \quad \ldots \quad (2.30)$$

where $A_4$, $B_4$ and $K^2$ are same as in (2.27) and

$$V(r) = -D_1 \cot^2 \alpha r \quad \ldots \quad \ldots \quad (2.31)$$

where $D_1$ and $K^2$ are same as in (2.29).

(B). The Stokes's differential equation [Pipes 1958, p. 355 (9.4)] which generally occurs in the theory of diffraction and refraction of waves as well as in the theory of transmission accross potential barriers, is given as

$$\frac{d^2w(z)}{dz^2}+bzw(z)=0 \qquad \ldots \quad \ldots \quad \ldots \quad (2.32)$$

whose solution [Pipes 1958, (9.4)] is

$$w(z)=\frac{z(M\sqrt{3}+N)(3b)^{1/6}}{\Gamma(\frac{1}{3})}\,{}_0F_1\left[-\,;\ \frac{-bz^3}{9}\right]+\frac{(-2N)}{(3b)^{1/6}\Gamma(2/3)}\,{}_0F_1\left[-\,;\ \frac{-bz^3}{9}\right] \qquad \ldots \quad (2.33)$$

where $M$ and $N$ are arbitrary constants.

Now, making the following substitutions

$$z=f(r),\ w(z)=g(r)\phi(r),\ h(r)=\frac{d}{dr}\{\log_e g(r)\}, \qquad \ldots \quad (2.34)$$

the eqn. (2.32) assumes the form of (2.6), where

$$A(r)=\frac{2g'(r)}{g(r)}-\frac{f''(r)}{f'(r)} \qquad \ldots \quad \ldots \quad \ldots \quad (2.35)$$

$$B(r)=\frac{g''(r)}{g(r)}-\frac{f''(r)g'(r)}{f'(r)g(r)}+bf(r)\{f'(r)\}^2 \qquad \ldots \quad (2.36)$$

and it will represent radial Schrödinger equ. (2.9), if the conditions of (2.10) are satisfied. In this way, we will be able to get the relation of (2.11) again and

$$h'(r)-h^2(r)+bf(r)\{f'(r)\}^2\equiv K^2-V(r). \qquad \ldots \quad \ldots \quad (2.37)$$

I. Coosing $z=f(r)=\gamma+e^{\alpha r}$ and taking the help of (2.11) and (2.37), the potential so obtained is

$$V(r)=-[b\gamma\alpha^2e^{-2\alpha r}+b\alpha^2e^{-3\alpha r}] \qquad \ldots \quad \ldots \quad \ldots \quad (2.38)$$

and

$$K^2=\frac{-\alpha^2}{4}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.39)$$

The result, thus derived in (2.38) is a two term exponential potential similar to the form of the well known Morse potential (Sneddon 1961, p. 176) and hence it may have useful application in nuclear motion of a diatomic molecule.

II. Proceeding as above, the considering

$$z=f(r)=\tanh\frac{\alpha r}{2}=\frac{e^{\alpha r}-1}{e^{\alpha r}+1}$$

we evaluate the potential

$$V(r) = -\left[\frac{4\alpha^2 b(e^{3\alpha r}-e^{2\alpha r})}{(1+e^{\alpha r})^5}\right] \quad \ldots \quad \ldots \quad (2.40)$$

and

$$K^2 = -\alpha^2/4. \quad \ldots \quad \ldots \quad (2.41)$$

III. Proceeding as before and choosing $z = f(r) = \gamma+\tan \alpha r$, the potential as constructed is

$$V(r) = -[\tan^4 \alpha r+\gamma \tan^3 \alpha r+2 \tan^2 \alpha r+2\gamma \tan \alpha r+1]b\alpha^2 \tan \alpha r \quad \ldots \quad (2.42)$$

and

$$K^2 = \alpha^2(1+b\gamma). \quad \ldots \quad \ldots \quad \ldots \quad (2.43)$$

IV. Similarly, for the choice $z = f(r) = 2\alpha r+\sin 2\alpha r$, we obtain the potential

$$V(r) = -\left[\frac{16\alpha^2 b \sin 2\alpha r-2\alpha^2 \tan^2 \alpha r \sec^4 \alpha r+32\alpha^3 br}{\sec^4 \alpha r}\right], \quad \ldots \quad (2.44)$$

and

$$K^2 = -\alpha^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.45)$$

V. In the same way, for the choice $z = f(r) = \gamma+\sin \alpha r$, we have the potential

$$V(r) = -\left[b\alpha^2(\gamma+\sin \alpha r)\cos^2 \alpha r-\frac{3\alpha^2}{4} \tan^2 \alpha r\right] \quad \ldots \quad (2.46)$$

and

$$K^2 = -\alpha^2/2. \quad \ldots \quad \ldots \quad \ldots \quad (2.47)$$

The potentials derived in (2.16), (2.18), (2.19), (2.38) and (2.40) are exponential potentials and rest of the potentials established here, are periodic potentials.

## 3. Bose's Method

The normal form of an ordinary second order differential equation can be written as

$$v''(\rho)+I(\rho)v(\rho) = 0. \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Putting $\rho = \rho(r)$ and $v(\rho) = (\rho')^{\frac{1}{2}} f(r)$ one gets

$$f''(r)+I_s(r)f(r) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where

$$I_s(r) = (\rho')^2 I(\rho)+\tfrac{1}{2}\{\rho, r\} \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

and $\{\rho, r\}$ stands for the Schwarzian derivative.

Now $I_s(r)$ of the $S$-wave radial Schrödinger equation is given by

$$I_s(r) = K^2 - V(r) \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

where $K^2$ is purely parametric term.

Hence, in order to solve the problem of the construction of potentials $V(r)$ for which the Schrödinger equation is solvable in terms of functions satisfying equ. (3.1) is the matter of determining transformation $\rho(r)$ such that the relation

$$(\rho')^2 I(\rho) + \tfrac{1}{2}\{\rho, r\} = K^2 - V(r) \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

holds.

The $I(\rho)$ of the generalized associated Legendre differential equ. (2.1) is given from (2.4) as

$$I(\rho) = \frac{4\lambda(\lambda+1)e^{2\rho} - m^2 e^{2\rho}(e^{2\rho}+1) - n^2(e^{2\rho}+1)}{(e^{2\rho}+1)^2}. \quad \ldots \quad (3.6)$$

Therefore, again for the choice of (2.13), using (3.3) and (3.6) we obtain

$I_s(r)$ = exactly the same expression on L.H.S. of (2.14), which, on using (3.5), determines exactly the same potential derived in (2.16).

Similarly, for the choices of (2.20) and (2.25) using (3.3), (3.6) and (3.5), we are led to exactly the same corresponding results (2.21), (2.22), (2.26) and (2.27).

Further, replacing $z$ by $\rho$ in Stokes's equ. (2.32), we have

$$w''(\rho) + b\rho w(\rho) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

which suggests that $I(\rho)$ of this Stokes's equ. is given as

$$I(\rho) = b\rho. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

Now, choosing $z = \rho = f(r) = \gamma + e^{-\alpha r}$ and using (3.3) and (3.8), we obtain

$$I_s(r) = b\gamma\alpha^2 e^{-2\alpha r} + b\alpha^2 e^{-3\alpha r} - \frac{\alpha^2}{4} \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

which, on using (3.5), furnishes exactly the same results (2.38) and (2.39).

Similarly, it can be shown that the potentials derived in (2.40), (2.42), (2.44) and (2.46) may easily be reconstructed with the help of this Bose's technique following the same corresponding transformations.

Thus, it has been obvious that for the same choice of transformations (value of $\rho$) similar potentials are established by the above two methods formulated for deriving solvable potentials.

## 4. Solutions

For the potential derived in (2.16), using (2.2), the solution $\phi(r)$ of the Schrödinger equ. (2.9) can be written as

$$\phi(r) = \sqrt{\frac{-2c}{\alpha}}\, e^{\frac{\alpha r}{4}}(1-e^{-\alpha r})^{\frac{1}{4}} A(e^{-\alpha r/2}-1)^{\frac{-m}{2}}\left(e^{\frac{-\alpha r}{2}}+1\right)^{\frac{n}{2}} \frac{1}{\Gamma(1-m)}$$

$$\times {}_2F_1\left[\begin{matrix} 1+\lambda-\frac{m}{2}+\frac{n}{2}, & -\lambda-\frac{m}{2}+\frac{n}{2}; \\ 1-m; \end{matrix} \quad \frac{1-e^{-\alpha r/2}}{2}\right]$$

$$+\sqrt{\frac{-2c}{\alpha}}\, e^{\alpha r/4}(1-e^{-\alpha r})^{\frac{1}{4}} B e^{\pi i m}(-1)^{\frac{2\lambda+1}{2}} 2^{\lambda+\frac{n}{2}-\frac{m}{2}}\Gamma\left(1+\lambda+\frac{m}{2}+\frac{n}{2}\right)$$

$$\times \frac{\Gamma\left(1+\lambda+\frac{m}{2}-\frac{n}{2}\right)}{\Gamma(2+2\lambda)}(1+e^{-\alpha_1/2})^{\frac{-2\lambda-2+m}{2}}\left(e^{-\alpha r/2}-1\right)^{\frac{-m}{2}}$$

$$\times {}_2F_1\left[\begin{matrix} \frac{2\lambda+n-m+2}{2}, & \frac{2\lambda-n-m+2}{2}; \\ 2\lambda+2; \end{matrix} \quad \frac{2}{1+e^{-\alpha r/2}}\right]. \qquad \ldots \quad (4.1)$$

From (2.15) $\alpha = \pm 4iK$ and thus, for determining the $S$-matrix explicitly, the values of $\alpha$ are chosen such that $\alpha = +4iK$ in $e^{\alpha r/4}$ in first term and $\alpha = -4iK$ in $e^{\alpha r/4}$ in the second term of the above solution (4.1).

Therefore, asymptotically $\phi(r)$ can be written as

$$\phi(r) \underset{r\to\infty}{\sim} \sqrt{\frac{-2c}{\alpha}}\, e^{ikr} A(-1)^{\frac{-m}{2}} \frac{1}{\Gamma(1-m)}$$

$$\times {}_2F_1\left[\begin{matrix} 1+\lambda-\frac{m}{2}+\frac{n}{2}, & -\lambda-\frac{m}{2}+\frac{n}{2}; \\ 1-m; \end{matrix} \quad \frac{1}{2}\right]$$

$$+\sqrt{\frac{-2c}{\alpha}}\, e^{-ikr} B e^{\pi i m}(-1)^{\frac{\lambda 2+1}{2}} 2^{\lambda+\frac{n}{2}-\frac{m}{2}}\Gamma\left(1+\lambda+\frac{m}{2}+\frac{n}{2}\right)$$

$$\times \frac{\Gamma\left(1+\lambda+\frac{m}{2}-\frac{n}{2}\right)}{\Gamma(2+2\lambda)}(-1)^{\frac{-m}{2}} {}_2F_1\left[\begin{matrix} \frac{2\lambda+n-m+2}{2}, & \frac{2\lambda-n-m+2}{2}; \\ 2\lambda+2; \end{matrix} \quad 2\right].$$

$$\ldots \quad (4.2)$$

Hence the $S$-matrix, in the explicit form is given as

$$S(K) = \frac{B\Gamma(1-m)e^{\pi i m}(-1)^{\frac{2\lambda+3}{2}}2^{\lambda+\frac{n}{2}-\frac{m}{2}}\Gamma\left(1+\lambda+\frac{m}{2}+\frac{n}{2}\right)}{A\Gamma(2+2\lambda)}$$

$$\times\Gamma\left(1+\lambda+\frac{m}{2}-\frac{n}{2}\right)\times\frac{{}_2F_1\left[\begin{matrix}\frac{2\lambda+n-m+2}{2},\frac{2\lambda-n-m+2}{2};\\ 2\lambda+2;\end{matrix}\ 2\right]}{{}_2F_1\left[\begin{matrix}1+\lambda-\frac{m}{2}+\frac{n}{2},-\lambda-\frac{m}{2}+\frac{n}{2};\\ 1-m;\end{matrix}\ \frac{1}{2}\right]}, \quad (4.3)$$

which is independent of $K$.

For the potential evaluated in (2.38), using (2.33), the solution $\phi(r)$ of the Schrödinger equ. (2.9) can be written as

$$\phi(r) = \sqrt{\frac{-c}{\alpha}}\,e^{\alpha r/2}\left\{\frac{(M\sqrt{3}+N)(3b)^{1/6}(\gamma+e^{-\alpha r}}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{4}{3};\end{matrix}\ \frac{-b(\gamma+e^{-\alpha r})^3}{9}\right]\right.$$

$$\left.+\frac{(-2)N}{(3b)^{1/6}\Gamma(2/3)}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{2}{3};\end{matrix}\ \frac{-b(\gamma+e^{-\alpha r})^3}{9}\right]\right\}. \quad \ldots \quad \ldots \quad (4.4)$$

Taking $\alpha = \pm 2iK$ from (2.39), we proceed similarly as before and hence, asymptotically $\phi(r)$ can be written as

$$\phi(r) \underset{r\to\infty}{\sim} \sqrt{\frac{-c}{\alpha}}\,e^{ikr}\,\frac{(M\sqrt{3}+N)(3b)^{1/6}\gamma}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{4}{3};\end{matrix}\ \frac{-b\gamma^3}{9}\right]$$

$$+\sqrt{\frac{-c}{\alpha}}\,e^{-ikr}\frac{(-2N)}{(3b)^{1/6}\,\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{2}{3};\end{matrix}\ \frac{-b\gamma^3}{9}\right]. \quad \ldots \quad \ldots \quad (4.5)$$

Hence the $S$-matrix, in the explicit form is given as

$$S(K) = \frac{2N\,\Gamma\left(\frac{1}{3}\right)}{\gamma(3b)^{1/3}\,\Gamma\left(\frac{2}{3}\right)(M\sqrt{3}+N)}\times\frac{{}_0F_1\left(-;\frac{2}{3};-b\gamma^3/9\right)}{{}_0F_1\left(-;\frac{4}{3};-b\gamma^3/9\right)}, \quad \ldots \quad (4.6)$$

which is independent of $k$.

## 5. Discussion

So far as the physical significance of the potentials derived in this paper, are concerned, the following may be mentioned :

(1) Potential derived in eqn. (2.18) can be used in the case of diatomic molecules since this satisfies the well known criterion for such molecules viz :

(I) Asymptotically the function $V(r)$ should be finite.

(II) $V(r)$ should be minimum at some distance $r = r_0$.

(III) $V(r)$ should be infinite for distance $r = 0$.

(2) The potentials derived in the equations (2.21), (2.26), (2.42), (2.44), (2.46) are all periodic potentials and thus may find application in solid state physics.

## Acknowledgements

The author is highly thankful to Dr. R. C. Varma for his kind supervision during the preparation of this paper. Further, he wishes to express his deep sense of gratitude to one of his colleagues Dr. L. K. Sharma for his ungrudged helping attitude and stimulating discussions to prepare this paper. He is indebted to the referee also for substantial improvements in the presentation of this paper.

## References

Aly, H. H., and Spector, R. M. (1965). Some solvable potentials for the Schrödinger equation. *Nuovo Cim.*, **38**, 149.

Bargmann, V. (1949). On the connection between phase shifts and scattering potentials. *Rev. mod. Phys.*, **21**, 488.

Bhattacharjie, A., and Sudershan, E. C. G. (1962). A class of solvable potentials. *Nuovo Cim.*, **25**, 864.

Bose, A. K. (1963). Solvable potentials. *Phys. Lett.*, **7**, 245.

Braaksma, B. L. J., and Meulenbeld, B. (1967). Integral transforms with generalized Legendre function's as Kernels. *Compositio math.*, 18, Fasc. **3**, 235–87.

Kuipers, L., and Meulenbeld, B. (1957). On a generalization of Legendre's associated differential equation I & II. *Proc. K. ned. Akad. Wet., Amstrerdam*, **60**, 436–50.

Manning, M. F., and Rosen, N. (1933). A potential function for the vibrations for diatomic molecules. *Phys. Rev.*, **44**, 953.

Newing, R. A. (1935). Note on the interrelation of the equilibrium nuclear distance with other molecular constants for diatomic colecules. *Phil. Mag.*, **19**, 759.

Newing, R. A. (1940). On the interrelation of molecular constants for diatomic molecules–II. *Phil. Mag.*, **29**, 298.

Pipes, L. A. (1958). Applied Mathematics for Engineers and Physicists. International Student Edition (Second Edition). McGraw-Hill Book Company, Inc., New York.

Sharma, L. K., and Varma, R. C. (1970). Derivation of a few solvable potentials for the Schrödinger equation. *Indian J. pure appl. Phys.*, **8**, No. 2, 66–69.

Sharma, L. K. (1970*a*). Potentials from modified Mathieu equation. *Proc. Indian natn. Sci. Acad.*, **36**, 230–38.

Sharma, L. K. (1970*b*). Solvable potentials from associated Legendre equation. *Proc. Indian natn. Sci. Acad.*, **36**, 239–45.

Sneddon, I. N. (1961). Special Functions of a Mathematical Physics and Chemistry, Second edition. Oliver & Boyd.