Indian J. pure appl. Math., 12(1) : 103–110, January 1981

A GENERALIZATION OF CERTAIN CLASSES OF POLYNOMIALS

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(*Received* 22 *February* 1980; *after revision* 7 *July* 1980)

In this paper we shall study a generalization of certain classes of polynomials by defining a Rodrigues' formula with the help of the operator $T_{a,k}$.

1. INTRODUCTION

Joshi and Prajapat (1975, 1977) have studied the class of polynomials defined by

$$M_{\nu n}^{(\alpha)}(x, a, k) = \frac{1}{n!} x^{-\alpha-nk} e^{p_\nu(x)} T_{a,k}^n \{x^\alpha e^{-p_\nu(x)}\} \qquad ...(1.1)$$

where $T_{a,k} = x^k(a + xD)$. It is clear that (1.1) may be regarded as the generalization of several systems of polynomials introduced earlier.

Recently, Chandel (1974) studied a generalization of several polynomial systems defined by

$$G_n(h, g, k) = e^{-hg}\Omega_x^n e^{hg}, \quad \Omega_x \equiv x^k \frac{d}{dx} \qquad ...(1.2)$$

where h and k are constants and g is function of x.

To unify the study of two general classes defined by (1.1) and (1.2) here we introduce the polynomials $G_n(a, k; h, g(x))$ defined by

$$G_n(a, k; h, g(x)) = e^{-hg(x)} T_{a,k}^n (e^{hg(x)}) \qquad ...(1.3)$$

where h is constant and $g(x)$ is differentiable function of x.

For $h = 1$ and $g(x) = \alpha \log_e x - p_\nu(x)$, $x > 0$ (1.3) is related to (1.1) by

$$G_n(a, k; 1, \alpha \log_e x - p_\nu(x)) = n!\, x^{kn} M_{\nu n}^{(\alpha)}(x, a, k) \qquad ...(1.4)$$

while for $a = 0$ and replacing k by $k - 1$ in (1.3), (1.3) is connected to (1.2) by

$$G_n(0, k - 1; h, g(x)) = G_n(h, g, k). \qquad ...(1.5)$$

Here we state the following known results which will be used in the paper:

$$F(T_{a,k})\{e^{g(x)}f(x)\} = e^{g(x)}F(T_{a,k} + x^{k+1}g'(x))\,f(x) \qquad \text{...(1.6)}$$

$$T^{n}_{a,k}\{x^{\lambda}uv\} = x^{\lambda}\sum_{n=0}^{\infty}\binom{n}{r}T^{n-r}_{a,k}(v)\,T^{r}_{a,k}(u) \qquad \text{...(1.7)}$$

$$T^{n}_{a,k} = x^{kn}\prod_{j=1}^{n}\{xD + a + (j-1)k\} \qquad \text{...(1.8)}$$

$$e^{tT_{a,k}}\{x^{\alpha}f(x)\} = \frac{x^{\alpha}}{(1 - x^{k}kt)^{(\alpha+a)/k}}\,f\left[\frac{x}{(1 - x^{k}kt)^{1/k}}\right] \qquad \text{...(1.9)}$$

$$a^{\delta}f(x) = f(ax),\ \delta \equiv xD \qquad \text{...(1.10)}$$

$$(1 + t)^{-\delta-\alpha}f(x) = \sum_{n=0}^{\infty}\frac{(-t)^{n}}{n!}(\delta + \alpha)_{n}\,f(x) \qquad \text{...(1.11)}$$

$$(1 + t)^{-\delta-\alpha} = (1 + t)^{-\delta}(1 + t)^{-\alpha}. \qquad \text{...(1.12)}$$

2. Some Operational Formulae

From (1.6) and (1.3) we get for $f(x) = 1$

$$G_{n}(a, k; h, g(x)) = (T_{a,k} + x^{k+1}hg'(x))^{n}.\ 1. \qquad \text{...(2.1)}$$

For brevity, let

$$T_{a,k} + x^{k+1}hg'(x) = \mathscr{D}. \qquad \text{...(2.2)}$$

Therefore

$$\mathscr{D}^{n}\{f(x)\} = e^{-hg(x)}T^{n}_{a,k}(e^{hg(x)}f(x)) \qquad \text{...(2.3)}$$

$$\mathscr{D}^{n}\{1\} = G_{n}(a, k; h, g(x)) \qquad \text{...(2.4)}$$

and

$$\mathscr{D}G_{n-1}(a, k; h, g(x)) = G_{n}(a, k; h, g(x)). \qquad \text{...(2.5)}$$

Again, using (2.3) and (1.7) we get

$$\mathscr{D}^{n}\{uv\} = \sum_{m=0}^{n}\binom{n}{m}T^{m}_{a,k}(u)\,\mathscr{D}^{n-m}(v) \qquad \text{...(2.6)}$$

which, for $v = 1$, gives

$$\mathscr{D}^{n}(u) = \sum_{m=0}^{n}\binom{n}{m}\mathscr{D}^{n-m}(1)\,T^{m}_{a,k}(u) \qquad \text{...(2.7)}$$

hence

$$\mathcal{D}^n = \sum_{m=0}^{n} \binom{n}{m} \mathcal{D}^{n-m}(1)\, T^m_{a,k}. \qquad \ldots(2.8)$$

Also

$$\mathcal{D}^n = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(a, k; h, g(x))\, T^m_{a,k}. \qquad \ldots(2.9)$$

Further, since

$$\mathcal{D}^{n+m} = \mathcal{D}^n \cdot \mathcal{D}^m \qquad \ldots(2.10)$$

we have

$$G_{n+m}(a, k; h, g(x)) = \mathcal{D}^m G_n(a, k; h, g(x)) = \mathcal{D}^n G_m(a, k; h, g(x)) \qquad \ldots(2.11)$$

which, on using (2.9), gives

$$G_{n+m}(a, k; h, g(x)) = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}(a, k; h, g(x))\, T^j_{a,k} \{G_m(a, k; h, g(x))\} \qquad \ldots(2.12)$$

An appeal to (1.7) and (2.4) shows that

$$T^n_{a,k} \{G_m(a, k; h, g(x))\} = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}(a, k; -h, g(x)) \times G_{m+j}(a, k; h, g(x)); \qquad \ldots(2.13)$$

if $m = 0$, then

$$T^n_{a,k} \{1\} = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}(a, k; -h, g(x))\, \mathcal{D}^j \{1\} \qquad \ldots(2.14)$$

from which we get the following results :

$$T^n_{a,k} = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}(a, k; -h, g(x))\, \mathcal{D}^j \qquad \ldots(2.15)$$

$${}_\lambda F_\mu \left[\begin{matrix} (a_\lambda); \\ \\ (b_\mu); \end{matrix}\; t\mathcal{D}\right] \{G_m(a, k; h, g(x))\} = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\lambda} (a_j)_n\, t^n}{\prod_{j=1}^{\mu} (b_j)_n} G_{n+m}(a, k; h, g(x)) \qquad \ldots(2.16)$$

$$(1 - t\mathscr{D})^{-1}\{1\} = \sum_{n=0}^{\infty} G_n(a, k; h, g(x))\, t^n; \; |t| \leqslant 1 \qquad ...(2.17)$$

$$(1 + t\mathscr{D})^n\{1\} = \sum_{j=0}^{n} \binom{n}{j} G_j(a, k; h, g(x))\, t^j \qquad ...(2.18)$$

$$L_n^{(\alpha)}(t\mathscr{D})\{1\} = \sum_{j=0}^{n} \frac{(1+\alpha)_n}{n!} \binom{n}{j} \frac{(-1)^j t^j}{(1+\alpha)_j} G_j(a, k; h, g(x)) \qquad ...(2.19)$$

and

$$(t + \mathscr{D})^n\{1\} = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}(a, k; h, g(x))\, t^j. \qquad ...(2.20)$$

Several other results can also be obtained as particular cases of (2.14).

3. Generating Functions and Recurrence Relations

By definition (1.3), we have

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\lambda} (a_j)_n\, t^n}{\prod_{j=1}^{\mu} (b_j)_n\, n!} G_n(a, k; h, g(x)) = e^{-hg(x)}\, {}_\lambda F_\mu \left[\begin{matrix} (a_\lambda); \\ (b_\mu); \end{matrix} \; tT_{a,k} \right] e^{hg(x)} \qquad ...(3.1)$$

which, for $\lambda = \mu$, $a_j = b_j$, $j = 1, 2, 3, \ldots, \lambda$ or μ, reduces to

$$\sum_{n=0}^{\infty} G_n(a, k; h, g(x)) \frac{t^n}{n!} = e^{-hg(x)} e^{tT_{a,k}} \{e^{hg(x)}\}. \qquad ...(3.2)$$

Now by an appeal to (1.9), the above result gives

$$\sum_{n=0}^{\infty} G_n(a, k; h, g(x)) \frac{t^n}{n!} = (1 - x^k kt)^{-a/k} \times \exp\left[h\left[g\left\{x(1 - x^k kt)^{-1/k}\right\} - g(x)\right]\right] \qquad ...(3.3)$$

in which, replacing t by $\frac{t}{kx^k}$, we get

$$\sum_{n=0}^{\infty} \frac{t^n G_n(a, k; h, g(x))}{n!\, k^n x^{kn}} = (1 - t)^{-a/k} \exp\left[h\left[g\left\{x(1 - t)^{-1/k}\right\} - g(x)\right]\right]. \qquad ...(3.4)$$

This result can also be obtained by using (1.10), (1.11) and (1.12) in the following way:

By an appeal to (1.8), (1.3) gives

$$\sum_{n=0}^{\infty} G_n(a, k; h, g(x)) \frac{t^n}{n!\, x^{nk}}$$

$$= e^{-hg(x)} \sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{j=0}^{n-1} (\delta + a + jk) \{e^{hg(x)}\}$$

$$= e^{-hg(x)} \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \left(\frac{\delta + a}{k}\right)_n \{e^{hg(x)}\}$$

$$= e^{-hg(x)}(1 - kt)^{-a/k} [(1 - kt)^{-1/k}]^{\delta} \{e^{hg(x)}\}$$

$$= e^{-hg(x)}(1 - kt)^{-a/k} e^{hg} \{x(1 - kt)^{-1/k}\}$$

$$= (1 - kt)^{-a/k} \exp [h [g \{x(1 - kt)^{-1/k}\} - g(x)]],$$

which on replacing t by t/k ultimately gives (3.4).

In a similar manner, considering

$$G_{n+m}(a, k; h, g(x)) = e^{-hg(x)} T_{a,k}^{n+m} e^{hg(x)}$$

we get

$$\sum_{n=0}^{\infty} \frac{G_{n+m}(a, k; h, g(x))\, t^n}{x^{kn}\, n!} = (1 - kt)^{-a/k} \exp [h [g \{x(1 - kt)^{-1/k}\} - g(x)]]$$

$$\times\ G_m(a, k; h, g \{x(1 - kt)^{-1/k}\}) \qquad \ldots(3.5)$$

which on replacing t by t/k, gives

$$\sum_{n=0}^{\infty} \frac{G_{n+m}(a, k; h, g(x))\, t^n}{k^n x^{kn}\, n!} = (1 - t)^{-a/k} \exp [h [g \{x(1 - t)^{-1/k}\} - g(x)]]$$

$$\times\ G_m(a, k; h, g \{x(1 - t)^{-1/k}\}). \qquad \ldots(3.6)$$

For $m = 0$ it yields (3.4).

Let $\phi_n(y)$ be polynomial of degree n in y given by

$$\phi_n(y) = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} \mu_j y^j \qquad \ldots(3.7)$$

where $\mu_j \neq 0$ are arbitrary constants, therefore in

$$\sum_{n=0}^{\infty} G_n(a, k; h, g(x))\, \phi_n(y)\, t^n = \sum_{n=0}^{\infty} G_n(a, k; h, g(x)) \frac{1}{n!} \times \sum_{j=0}^{n} \binom{n}{j} \mu_j y^j t^n$$

following Srivastava and Singhal (1971) and making an appeal to (3.6) we get a bilateral generating function

$$\sum_{n=0}^{\infty} G_n(a, k; h, g(x))\, \phi_n(y)\, t^n$$

$$= (1 - tkx^k)^{-a/k} \exp [h [g \{x(1 - tkx^k)^{-1/k}\} - g(x)]] \times F(x(1 - tkx^k)^{-1/k}, yt) \qquad \ldots(3.8)$$

where

$$F(x, t) = \sum_{j=0}^{\infty} \frac{1}{j!} \mu_j G_j(a, k; h, g(x))\, t^j. \qquad \ldots(3.9)$$

Starting with the generating relation (3.4) we derive

$$G_n(a, k; h, g(x)) = \sum_{m=0}^{n} \left(\frac{a-b}{k}\right)_m \binom{n}{m} (kx)^{mk}\, G_{n-m}(b, k; h, g(x)). \qquad \ldots(3.10)$$

For $m = 1$ (2.11) gives a recurrence relation

$$\{a + xhg'(x)\}\, x^k G_n(a, k; h, g(x))$$

$$= G_{n+1}(a, k; h, g(x)) - x^{k+1} D G_n(a, k; h, g(x)). \qquad \ldots(3.11)$$

4. An Extension of (2.9) and (2.15)

From (2.9) we have

$$\underset{h, g(x)}{\overset{n}{\mathcal{D}}} = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(a, k; h, g(x))\, T^m_{a,k}$$

and from (2.15), we have $T^n_{a,k} = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}(a, k; -h, g(x)) \underset{h, g(x)}{\overset{j}{\mathcal{D}}}$. Now combining above two results, we obtain

$$\mathop{\mathscr{D}}_{h,g(x)}^{n} = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(a, k; h, g(x))$$

$$\times \sum_{j=0}^{m} \binom{m}{j} G_{m-j}(a, k; -l, f(x)) \mathop{\mathscr{D}}_{l,f(x)}^{j} \qquad \text{...(4.1)}$$

$$= \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(a, k; h, g(x))$$

$$\times \sum_{j=0}^{m} \binom{m}{j} G_{m-j}(a, k; -l, g(x)) \mathop{\mathscr{D}}_{l,g(x)}^{j} \qquad \text{...(4.2)}$$

$$= \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(a, k; h, g(x))$$

$$\times \sum_{j=0}^{m} \binom{m}{j} G_{m-j}(a, k; -h, f(x)) \mathop{\mathscr{D}}_{h,f(x)}^{j} \qquad \text{...(4.3)}$$

and

$$\mathop{\mathscr{D}}_{h,g(x)}^{n} = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(a, k; h, g(x))$$

$$\times \sum_{j=0}^{m} \binom{m}{j} G_{m-j}(a, k; h, -f(x)) \mathop{\mathscr{D}}_{h,f(x)}^{j} \cdot \qquad \text{...(4.4)}$$

5. A Wronskian for $G_n(a, k; h, g(x))$

Recently, Chandel (1974) gave a Wronskian for $G_n(h, g; k)$. On the same lines we can also derive the following Wronskian for $G_n(a, k; h, g(x))$.

Theorem — If

$$\sigma_n(a, k; h, g(x)) = \text{the det. of } [G_{n+i-j}(a, k; h, g(x))], \quad 0 \leqslant i, j \leqslant m \qquad \text{...(5.1)}$$

and

$$W\{G_n(a, k; h, g(x)), G_{n-1}(a, k; h, g(x)), \ldots, G_{n-m}(a, k; h, g(x))\}$$

$$= \text{the det. of } [T^{i}_{a,k} G_{n-j}(a, k; h, g(x))], \quad 0 \leqslant i, j \leqslant m \qquad \text{...(5.2)}$$

then

$$\sigma_n(a, k; h, g(x)) = W\{G_n(a, k; h, g(x)), G_{n-1}(a, k; h, g(x)), \ldots, G_{n-m}(a, k; h, g(x))\}. \quad \ldots(5.3)$$

ACKNOWLEDGEMENT

The authors are thankful to the referee for his valuable suggestions.

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