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HYDRODYNAMIC FLOW PAST AN ACCELERATED POROUS PLATE IN ROTATING SYSTEM

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An initial value investigation is made of the motion of an incompressible, homogeneous, viscous fluid over a porous plate with uniform suction or blowing. Both the plate and the fluid are in state of solid body rotation with constant angular velocity about z-axis normal to the plate and the plate is assumed to be accelerated with a given velocity. A solution describing the general feature of the unsteady hydrodynamic boundary layer flow in a rotating fluid with suction (blowing) has been obtained. The results obtained have been compared with those of previous works. Some particular cases of interest have been discussed. Special attention has been given to the physical interpretation of mathematical results obtained for steady state solution. The significant effect of the suction parameter on the flow phenomenon in this case has also been investigated.

1. INTRODUCTION

Gupta (1972) has studied an exact solution for the steady state three dimensional Navier-Stokes equation for the flow past a plate with uniform suction in a rotating coordinate system. Debnath and Mukherjee (1973) have studied the motion of an incompressible homogeneous, viscous fluid bounded by porous plates with uniform suction (blowing). Both the fluid and the plates are in a state of solid body rotation with constant angular velocity Ω about z-axis normal to the plate and additionally a non-torsional oscillation of a given frequency ω is imposed on the plate for generation of unsteady flow in a rotating system.

In the present paper, the authors have reviewed problem of Debnath and Mukherjee (1973) under different boundary conditions. In our case the plate is assumed to be accelerated with a velocity $Ae^{i\omega t}t^m$ (A and m being the constants). Debnath and Mukherjee have considered elliptic harmonic oscillations of the plate.

2. MATHEMATICAL FORMULATIONS

The unsteady motion of a viscous fluid in a rotating coordinate system is governed by the Navier-Stokes equation and the continuity equation (Debnath 1972)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\,\nabla)\,\mathbf{u} + 2\Omega\mathbf{k} \times \mathbf{u} = -\rho^{-1}\,\nabla p + \nu\,\nabla^2\,\mathbf{u} \qquad ...(2.1)$$

$$\text{div}\,\mathbf{u} = 0, \qquad \text{...(2.2)}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector, $\mathbf{k}$ the unit vector along z-axis, p the pressure including centrifugal term, ρ the density and ν the kinematic viscosity. We assume that the velocity field depends on z and t alone, so that

$$\mathbf{u}(z, t) = [u(z, t), v(z, t), w(z, t)]. \qquad \text{...(2.3)}$$

It follows from eqn. (2.2), together with uniform suction (blowing), that $w = -w_0$ is constant. Obviously $w_0 > 0$ for suction and $w_0 < 0$ for blowing.

In the absence of pressure gradient, the equation of motion (2.1) takes the form

$$\frac{\partial u}{\partial t} - w_0 \frac{\partial u}{\partial z} - 2\Omega v = \nu \frac{\partial^2 u}{\partial z^2} \qquad \text{...(2.4)}$$

$$\frac{\partial v}{\partial t} - w_0 \frac{\partial v}{\partial z} + 2\Omega u = \nu \frac{\partial^2 v}{\partial z^2}. \qquad \text{...(2.5)}$$

Introducing the notation $q = u + iv$, eqns. (2.4) and (2.5) become

$$\frac{\partial q}{\partial t} - w_0 \frac{\partial q}{\partial z} + 2\Omega i q = \nu \frac{\partial^2 q}{\partial z^2}. \qquad \text{...(2.6)}$$

The boundary conditions for the problem are

$$q(z, t) = 0 \text{ for all } z = 0 \text{ and } t \leqslant 0, \qquad \text{...(2.7)}$$

$$q(z, t) = Ae^{i\omega t}t^m,\ w = w_0 \text{ for } z = 0 \text{ and } t > 0, \qquad \text{...(2.8)}$$

and

$$q \to 0 \text{ or finite as } z \to \infty \text{ for } t > 0, \qquad \text{...(2.9)}$$

where A is constant with the dimension of velocity. The present boundary conditions are generalization of Debnath and Mukherjee (1973) which appear to have some physical significance for the understanding of geophysical and astrophysical problems and is likely to reveal some important and interesting characteristic features of the hydrodynamic spin up flows as pointed out by Debnath (1975).

3. Solution of the Problem

It is convenient to introduce the non-dimensional variables

$$z' = \frac{zA}{\nu}, \quad t' = \Omega t, \quad q' = q/A \qquad \text{...(2.10)}$$

and non-dimensional parameters

$$S = \frac{w_0}{A}, \quad E = \frac{2\Omega\nu}{A^2}, \quad \sigma = \frac{\omega}{\Omega}. \qquad \text{...(2.11)}$$

In terms of there non-dimensional quantities, eqn. (2.6) and the boundary and initial conditions (2.7)–(2.9) become, on dropping the primes

$$\frac{\partial^2 q}{\partial z^2} + S\frac{\partial q}{\partial z} - iEq = \frac{E}{2}\frac{\partial q}{\partial t} \qquad \text{...(2.12)}$$

$$q = 0 \text{ everywhere for } t \leqslant 0 \qquad \text{...(2.13)}$$

$$q = Ae^{i\sigma t}t^m \text{ at } z = 0,\ t > 0 \qquad \text{...(2.14)}$$

and

$$q = 0 \text{ or finite as } z \to \infty,\ t > 0. \qquad \text{...(2.15)}$$

In order to solve the initial value problem, we introduce the Laplace transform $\bar{q}(z, p)$ of $q(z, t)$ defined by the integral

$$\bar{q}(z, p) = \int_0^\infty e^{-pt} q(z, t)\, dt. \qquad \text{...(2.16)}$$

The Laplace transform of eqn. (2.12) and the boundary conditions (2.13) – (2.15) are given by

$$\frac{d^2\bar{q}}{dz^2} + S\frac{d\bar{q}}{dz} - \left(iE + \frac{pE}{2}\right)\bar{q} = 0 \qquad \text{...(2.17)}$$

$$\bar{q}(z, p) = \frac{A\Gamma(m+1)}{(p - i\sigma)^{m+1}} \text{ at } z = 0 \qquad \text{...(2.18)}$$

$$\bar{q}(z, p) = 0 \text{ or finite as } z \to \infty. \qquad \text{...(2.19)}$$

Solving eqn. (2.17) with the boundary conditions (2.18), (2.19), we get

$$\bar{q}(z, p) = \frac{A\Gamma(m+1)}{(p - i\sigma)^{m+1}} \exp\left[\frac{-zS}{2} - z\sqrt{\frac{E}{2}}\left(p + \frac{S^2 + 4iE}{2E}\right)^{1/2}\right]. \qquad \text{...(2.20)}$$

Inverting eqn. (2.20) by Fourier-Mellin inversion integral, we get

$$q(z, t) = \frac{A\Gamma(m+1)}{2\pi i} \exp\left(\frac{-zS}{2}\right) \int_{\lambda - i\infty}^{\lambda + i\infty} \exp\left[pt - z\sqrt{\frac{E}{2}} \times \left(p + \frac{S^2 + 4iE}{2E}\right)^{1/2}\right] \frac{dp}{(p - i\sigma)^{m+1}} \qquad \text{...(2.21)}$$

Let us put

$$C = z\sqrt{\frac{E}{2}},\quad x^2 = \left(p + \frac{S^2 + 4iE}{2E}\right),\quad \alpha = \left(i\sigma + \frac{S^2 + 4iE}{2E}\right).$$

Then we have

$$q(z, t) = A\,\Gamma(m+1)\,\exp\left(\frac{-zS}{2}\right) I(z, t, \alpha, m) \qquad \ldots(2.22)$$

where

$$I(z, t, \alpha, m) = \frac{1}{2\pi i}\int_{Br_3} \exp\left[\left\{x^2 - \frac{S^2+4iE}{2E}\right\} t - Cx\right] \frac{2x}{(x^2-\alpha)^{m+1}}\,dx \qquad \ldots(2.23)$$

the path Br_3 is Bromwich path defined in McLachlan (1963).

Now we have to find the values of $I(z, t, \alpha, m)$ for different values of m. Let

$$F(\alpha) = I(z, t, \alpha, 0) = \frac{1}{2\pi i}\int_{Br_3} \exp\left[\left\{x^2 - \frac{S^2+4iE}{2E}\right\} t - Cx\right] \times \frac{2x}{(x^2-\alpha)}\,dx. \qquad \ldots(2.24)$$

Differentiating eqn. (2.24) with respect to α, we get

$$I(z, t, \alpha, 1) = \frac{dF}{d\alpha} + tF. \qquad \ldots(2.25)$$

Again differentiating eqn. (2.25) successively $(m-1)$ times with respect to α, we get $I(z, t, \alpha, m)$ for different values of m.

Using McLachlan (1963), we have

$$I(z, t, \alpha, 0) = \tfrac{1}{2}\exp(i\sigma t)\left[\exp(C\sqrt{\alpha})\,\mathrm{erfc}\left(\frac{C+2t\sqrt{\alpha}}{2\sqrt{t}}\right) + \exp(-C\sqrt{\alpha})\,\mathrm{erfc}\left(\frac{C-2t\sqrt{\alpha}}{2\sqrt{t}}\right)\right]. \qquad \ldots(2.26)$$

With the help of eqns. (2.25) and (2.26), we get

$$I(z, t, \alpha, 1) = \tfrac{1}{2}\exp(i\sigma t)\left[\exp(C\sqrt{\alpha})\,\mathrm{erfc}\left(\frac{C+2t\sqrt{\alpha}}{2\sqrt{t}}\right)\left(\frac{C}{2\sqrt{\alpha}}+1\right) - \exp(C\sqrt{\alpha})\,\mathrm{erfc}\,\frac{(C-2t\sqrt{\alpha})}{2\sqrt{t}}\left(\frac{C}{2\sqrt{\alpha}}-t\right)\right]. \qquad \ldots(2.27)$$

Similarly we can get $I(z, t, \alpha, 2)$ and in general $I(z, t, \alpha, m)$ for different integral values of m.

Particular cases

Case I: *When* $i\sigma t \neq 0$ *and* $m = 0$ — The velocity field is obtained with the help of eqns. (2.22) and (2.26) as

$$\phi = \frac{q(z,t)}{A} = \tfrac{1}{2}\exp\left[i\sigma t - \frac{Sz}{2}\right]\left[\exp\left\{z\left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}\right.$$
$$\times \operatorname{erfc}\left\{z\left(\frac{E}{2}\right)^{1/2} + \left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2} t^{1/2}\right\}$$
$$+ \exp\left\{-z\left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}$$
$$\left.\times \operatorname{erfc}\left\{\frac{z}{2}\left(\frac{E}{2t}\right)^{1/2} - \left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2} t^{1/2}\right\}\right] \qquad ...(2.28)$$

where erfc (x) is the complementary error function defined by Lebedev (1965)

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\pi}\int_x^\infty e^{-\tau^2}\, d\tau. \qquad ...(2.29)$$

The solution (2.28) describes the general feature of the unsteady hydrodynamic boundary layer flow in a rotating fluid including the effect of uniform suction (blowing) according as $S > 0$ or $S < 0$. This result agrees with that of Gupta (1972) and Debnath and Mukherjee (1973). By setting $S = 0$, the velocity distribution (2.28) is exactly identical with that of Thornely (1968).

Case II: *When* $i\sigma t \neq 0$ *and* $m = 1$ — The velocity field obtained with the help of eqns. (2.22) and (2.27) is

$$\phi = \frac{q(z,t)}{At} = \tfrac{1}{2}\exp\left[i\sigma t - \frac{Sz}{2}\right]\left[\exp\left\{z\left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}\right.$$
$$\times \operatorname{erfc}\left\{\frac{z}{2}\left(\frac{E}{2t}\right)^{1/2} + \left(i\sigma + \frac{S^2+4iE}{2E}\right)\right\}\left\{z\left(\frac{E}{2}\right)^{1/2}\right.$$
$$\left.\times \frac{1}{2\left[\left(i\sigma + \frac{S^2+4iE}{2E}\right)t\right]^{1/2}} + 1\right\} - \exp\left\{-z\left(\frac{E}{2}\right)^{1/2}\right.$$
$$\left.\times\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}\operatorname{erfc}\left\{\frac{z}{2}\left(\frac{E}{2t}\right)^{1/2} - \left(i\sigma + \frac{S^2+4iE}{2E}\right)\right\}$$
$$\left.\times\left\{\frac{z}{2}\left(\frac{E}{2}\right)^{1/2}\left[\left(i\sigma + \frac{S^2+4iE}{2E}\right)t\right]^{-(1/2)} - 1\right\}\right]. \qquad ...(2.30)$$

Case III: *When* $i\sigma t \neq 0$ *and* $m = 2$ — With the help of eqns. (2.22) and (2.25) we have

$$\phi = q(z,t) = A\exp\left(\frac{-zS}{2}\right)\left[\frac{d}{d\alpha} I(z,t,\alpha,1) + tI(z,t,\alpha,1)\right]. \qquad ...(2.31)$$

Using eqns. (2.27) and (2.31) the velocity field is obtained as

$$\phi = \frac{q(z,t)}{At^2} = \tfrac{1}{2}\exp\left[i\sigma t - \frac{Sz}{2}\right]\left[\exp\left\{z\left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}\right.$$

$$\times \operatorname{erfc}\left\{\frac{z}{2}\left(\frac{E}{2t}\right)^{1/2} + \left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2} t^{1/2}\right\}$$

$$\times \left\{\frac{z^2E}{8\left(i\sigma + \frac{S^2+4iE}{2E}\right)} + \frac{z\sqrt{E}}{\sqrt{2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}} - \frac{z\sqrt{E}}{4\sqrt{2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{3/2}} + 1\right\}$$

$$+ \exp\left\{-z\left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}$$

$$\times \operatorname{erfc}\left\{z\left(\frac{E}{2t}\right)^{1/2} - \left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2} . \, t^{1/2}\right\}$$

$$\times \left\{\frac{z^2E}{8\left(i\sigma + \frac{S^2+4iE}{2E}\right)} - \frac{z\sqrt{E}}{\sqrt{2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}} + \frac{z\sqrt{E}}{u\sqrt{2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}} + 1\right\}$$

$$\left. - \frac{z\sqrt{E}}{\pi\sqrt{2}}\exp\left\{\frac{-z^2E}{8} + \left(i\sigma + \frac{S^2+4iE}{2E}\right)\right\}\right]. \quad \ldots(2.32)$$

By asymptotic representation of the complementary error function as $t \to \infty$ the solution (2.28) reduces to the steady state form

$$\phi = \frac{q(z,t)}{A} = \exp\left[i\sigma t - z\left\{\frac{S}{2} + \left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}\right]. \quad \ldots(2.33)$$

which includes the effect of uniform suction.

By setting $S = -S_1$, it follows that $S_1 > 0$ and the result (2.33) reduces to the form

$$\phi = \frac{q(z,t)}{A} = \exp\left[i\sigma t - z\left\{\frac{-S_1}{2} + \left(\frac{E}{2}\right)^{1/2}\left(i\sigma + \frac{S^2+4iE}{2E}\right)^{1/2}\right\}\right] \quad \ldots(2.34)$$

including the effect of uniform blowing.

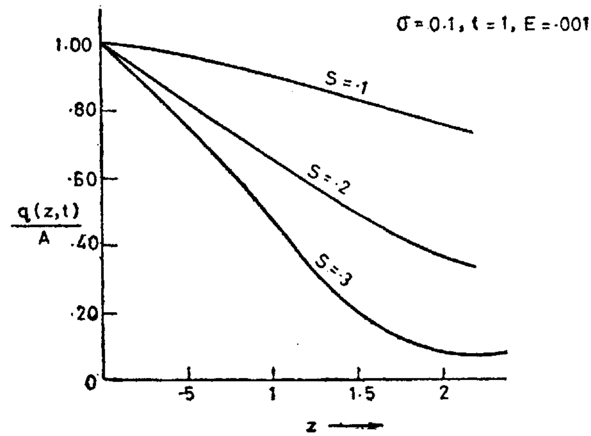

FIG. 1. Velocity distribution $q(z, t)/A$ against z.

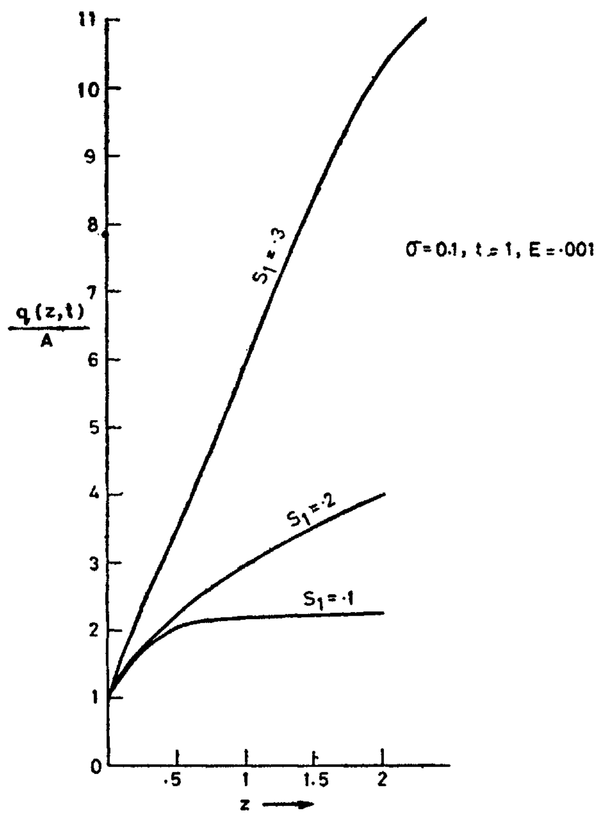

FIG. 2. Velocity distribution $q(z, t)/A$ against z.

Concluding Remarks

The solution given by eqns. (2.22) and (2.23) describes the general feature of the unsteady hydrodynamic boundary layer flow in a rotating system. In particular for $m = 0$, we have a pure oscillation of given frequency ω imposed on the plate and the corresponding solution which we have obtained is in close agreement with Debnath (1973). Equations (2.30) and (2.32) give the velocity distributions for special cases when $m = 1$ and 2 respectively. In particular, the steady state solution has been obtained when $i\sigma t \neq 0$ and $m = 0$. The effects of suction and blowing on the steady state solution has been explained graphically.

Figure 1 displays the effect of suction on the steady state velocity profile. Here we have plotted $q(z, t)/A$ against z for different values of suction parameter $S = 0.1$, 0.2 and 0.3 respectively. It is noticed that velocity decreases more rapidly with increase in suction parameter and consequently the boundary layer thickness is reduced with the increase in suction parameter. Figure 2 shows the effect of blowing on the flow for same values of σ, t and E ($\sigma = 0.1$, $t = 1$, $E = 0.001$). It has been plotted for different values of blowing parameter S_1 ($S_1 = 0.1$, 0.2 and 0.3 respectively). It is concluded that the blowing has a growing effect on the velocity profile and the velocity increases rapidly with the blowing parameter S_1.

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