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ON SOME TRANSFORMATIONS OF THE GENERALIZED ISOTHERMAL EQUATION(GIE) AND PADÉ APPROXIMANTS

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We have studied a few transformations which connect solutions of the "generalized" isothermal equation (analogue of the Lane-Emden equation) in $(u_\rho;\ v_\rho)$-, $(u_P;\ v_P)$-, $(u_\psi;\ v_\psi)$-planes and $(y_\rho;\ z_\rho)$-, $(y_P;\ z_P)$-, $(y_\psi;\ z_\psi)$-planes. Approximate analytical solutions for the spherical and plane-symmetric cases in $(u_\psi;\ v_\psi$-plane are tabulated following the technique of Padé approximants. Physical interpretations of the results have also been included.

1. INTRODUCTION

Several authors have discussed the theory of self-gravitating isothermal gas spheres (classical) (Emden 1907, Eddington 1926, Milne 1930, Chandrasekhar 1939, Srivastava 1976). Ostriker (1964a, b) has studied the equilibrium of a self-gravitating isothermal cylindrical configuration. Spitzer (1942), Ledoux (1951), Goldreich and Lynden-Bell (1965) have made their valuable contibutions towards a detailed study of the isothermal sheets (Saturn ring system and the Laplacian-disk cosmogonies). The density distribution in any region of these non-rotating configurations is governed by the "generalized" isothermal equation (GIE):

$$\frac{1}{\xi_\psi^m}\frac{d}{d\xi_\psi}\xi_\psi^m\frac{d\psi}{d\xi_\psi}=e^{-\psi} \qquad \ldots(1)$$

an equation analogous to the Lane-Emden equation. Equation (1) satisfies the following boundary conditions

$$\psi=0;\quad \psi'=0 \text{ at } \xi_\psi=0, \left(\psi'=\frac{d\psi}{d\xi_\psi}\right) \qquad \ldots(2)$$

and the dimensionless variables ξ and ψ are defined by

$$\rho\equiv\lambda e^{-\psi};\qquad r\equiv\left[\frac{K}{4\pi G\lambda}\right]^{1/2}\xi_\psi\equiv\alpha\xi_\psi\qquad\left(K=\frac{k}{\mu H}T\right). \qquad \ldots(3)$$

Plane-symmetric, cylindrical and spherical configurations are given by the values of $m=0$, 1 and 2, respectively. G, λ and ρ denote, respectively, the gravitational

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constant (= 6.67×10^{-8} dynes cm^2/gm^2), the central density (for a complete isothermal gas configuration) and density. r represents the radius, the length along the axis and the thickness, respectively, for spherical, cylindrical and plane-symmetric cases. Classical study of eqn. (1) has generally been made in (ξ_ψ; ψ)-plane. In order that (like polytropes) (see Srivastava and Sharma 1969; Sharma 1970, 1973, Srivastava 1977) the study of isothermal equation (1) may prove useful in other planes also, we have derived the generalized first-order differential equations, equivalent to (1), in (u_P; v_P)-, (u_ρ; v_ρ)-, (u_ψ; v_ψ)-planes and (y_P; z_P)-, (y_ρ; z_ρ)-, (y_ψ; z_ψ)*-planes (section 2). The transformation-formulae connecting the solutions from one plane to another are given in section 3. As an illustration to the application of one of the first-order differential equations to our physical problems, we present in section 4 approximate analytical solutions of eqn. (5) for plane-symmetric ($m = 0$) and spherical ($m = 2$) configurations (Table I) [so far the solution for the case $m = 2$ in (u_ψ; v_ψ)-plane is available in numerical form (Chandrasekhar 1939) only]. Suffixes denote the variables in which the equations of hydrostatic equilibrium have been expressed, for example, suffix P means we have cosidered the isothermal equation in (r, P)-plane and (u_P; v_P) and (y_P; z_P) are the variables which transform the fundamental equation into the first-order equations. This is also true for the suffixes ρ and θ.

2. Generalized First-order Differential Equations

(a) The Generalized First-order Differential Equations in (u_ψ; v_ψ)-, (u_P; v_P)-, and (u_ρ; v_ρ)-planes

Consider the two functions u_ψ and v_ψ related with the variables ξ_ψ and ψ by equations

$$u_\psi = \frac{\xi_\psi e^{-\psi}}{\psi'}, \quad v_\psi = \xi_\psi \psi' \qquad \ldots(4)$$

then the generalized isothermal equation (1) is reduced to the first-order differential equation:

$$\frac{u_\psi}{v_\psi}\frac{dv_\psi}{du_\psi} = \frac{1 + u_\psi - m}{1 - u_\psi - v_\psi + m}. \qquad \ldots(5)$$

The generalized forms of the fundamental isothermal equations in (r, P)- and (r, ρ)-variables are

$$\frac{K^2}{4\pi G}\frac{1}{r^m}\frac{d}{dr}\left(\frac{r^m}{P}\frac{dP}{dr}\right) = -P \qquad \ldots(6)$$

*The importance of the use of (y_ψ; z_ψ)-variables in reducing (1) to the first-order differential equation was pointed out by Emden (1907) for the first time.

and

$$\frac{K}{4\pi G}\frac{1}{r^m}\frac{d}{dr}\left(\frac{r^m}{\rho}\frac{d\rho}{dr}\right) = -\rho \qquad \text{...(7)}$$

respectively, which can be re-written in the form

$$\frac{1}{\xi_P^m}\frac{d}{d\xi_P}\left(\frac{\xi_P^m}{P}\frac{dP}{d\xi_P}\right) = -P \qquad \text{...(6a)}$$

and

$$\frac{1}{\xi_\rho^m}\frac{d}{d\xi_\rho}\left(\frac{\xi_\rho^m}{\rho}\frac{d\rho}{d\xi_\rho}\right) = -\rho \qquad \text{...(7a)}$$

where the dimensionless variables ξ_P and ξ_ρ are defined by

$$r \equiv \alpha_P \xi_P \equiv \frac{K}{\sqrt{4\pi G}}\xi_P \qquad \text{...(8)}$$

and

$$r \equiv \alpha_\rho \xi_\rho \equiv [K/4\pi G]^{1/2}\,\xi_\rho. \qquad \text{...(9)}$$

The four variables u_P and v_P; u_ρ and v_ρ defined by the equations

$$u_P = \frac{\xi_P P^2}{P'},\quad v_P = \frac{\xi_P P'}{P}\quad \left(P' = \frac{dP}{d\xi_P}\right) \qquad \text{...(10)}$$

$$u_\rho = \frac{\xi_\rho \rho^2}{\rho'},\quad v_\rho = \frac{\xi_\rho \rho'}{\rho}\quad \left(\rho' = \frac{d\rho}{d\xi_\rho}\right) \qquad \text{...(11)}$$

transform (6*a*) and (7*a*) into the following first-order differential equations

$$\frac{u_P}{v_P}\frac{dv_P}{du_P} = \frac{1 - u_P - m}{1 + u_P + v_P + m} \qquad \text{...(12)}$$

and

$$\frac{u_\rho}{v_\rho}\frac{dv_\rho}{du_\rho} = \frac{1 - u_\rho - m}{1 + u_\rho + v_\rho + m} \qquad \text{...(13)}$$

respectively. On putting $m = 0, 1$ and 2 in each of eqns. (5), (12) and (13) we obtain particular expressions for plane-symmetric, cylindrical and spherical* configurations.

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*Chandrasekhar has derived the first-order equation in $(\xi_\psi;\ \psi)$-plane for the spherical case $(m = 2)$ only.

(*b*) *The Generalized First-Order Differential Equations in* $(y_\psi; z_\psi)$-, $(y_P; z_P)$- *and* $(y_\rho; z_\rho)$-*planes*

The relations between the variables $(z_\psi; y_\psi)$ and $(\xi_\psi; \psi)$ are given by

$$z_\psi = -\psi + 2\log\xi_\psi \qquad \text{...(14)}$$

and
$$y_\psi = \frac{dz_\psi}{dt_\psi} = \xi_\psi \frac{d\psi}{d\xi_\psi} - 2;\ \xi_\psi = e^{-t_\psi}. \qquad \text{...(15)}$$

Equations (14) and (15) enable us to reduce (1) to the form of generalized first-order differential equation:

$$y_\psi \frac{dy_\psi}{dz_\psi} + (1-m)\,y_\psi + 2\,(1-m) + e^z = 0,\ (m \neq 1) \qquad \text{...(16)}$$

which, on putting $m = 2$, leads to give Chandrasekhar's equation. If we further define

$$\left.\begin{array}{ll} \text{(a)} & z_P = \xi_P^{-\omega_P} P;\ \omega_P = -2 \\ \text{(b)} & y_P = \dfrac{dz_P}{dt_P} = -\,\xi_P^{-\omega_P+1}\dfrac{dP}{d\xi_P} + \omega_P z_P;\ \xi_P = e^{-t_P} \end{array}\right\} \qquad \text{...(17)}$$

and

$$\left.\begin{array}{ll} \text{(a)} & z_\rho = \xi_\rho^{-\omega_\rho} \rho;\ \omega_\rho = -2 \\ \text{(b)} & y_\rho = \dfrac{dz_\rho}{dt_\rho} = -\xi_\rho^{-\omega_\rho+1}\dfrac{d\rho}{d\xi_\rho} + \omega_\rho z_\rho;\ \xi_\rho = e^{-t_\rho} \end{array}\right\} \qquad \text{...(18)}$$

then (6*a*) and (7*a*) are transformed into two similar first-order differential equations

$$y_P \frac{dy_P}{dz_P} - y_P^2\, z_P^{-1} + (1-m)\,y_P + (m-1)\,\omega_P z_P + z_P^2 = 0,\ (m \neq 1) \qquad \text{...(19)}$$

and
$$y_\rho \frac{dy_\rho}{dz_\rho} - y_\rho^2\, z_\rho^{-1} + (1-m)\,y_\rho + (m-1)\,\omega_\rho z_\rho + z_\rho^2 = 0,\ (m \neq 1) \qquad \text{...(20)}$$

respectively. Similarity between (19) and (20) is due to the transformations (17) and (18), as applied to the two analogous expressions (6*a*) and (7*a*).

(3) Transformations Connecting Solutions of Isothermal Equations in $(u_\psi; v_\psi)$-, $(u_P; v_P)$-, $(u_\rho; v_\rho)$-Planes; and $(z_\psi; y_\psi)$-, $(z_P; y_P)$-, $(z_\rho; y_\rho)$-Planes

Dividing first equation of (4) by the first equation of (10) and using (3) and (8), we get

$$\frac{u_\psi}{u_P} = \frac{\alpha_P}{\alpha_\psi} \frac{e^{-\psi}P'}{\psi' P^2}. \qquad ...(21)$$

Substituting

$$P' = -K\lambda \frac{\alpha_P}{\alpha_\psi} e^{-\psi}\psi' \qquad ...(22)$$

in the foregoing equation, we find

$$u_\psi = - u_P. \qquad ...(23)$$

Further, dividing first equation of (4) by first equation of (11); using (3), (9) and the relation

$$\rho' = -\lambda \frac{\alpha_\rho}{\alpha_\psi} e^{-\psi}\psi' \qquad ...(24)$$

we obtain

$$u_\psi = -u_\rho. \qquad ...(25)$$

We deduce from (23) and (25),

$$u_P = u_\rho. \qquad ...(26)$$

Proceeding exactly in a similar way as above, we easily find that

$$\left.\begin{aligned} v_\psi &= -v_P \\ v_\psi &= -v_\rho \\ v_P &= v_\rho. \end{aligned}\right\}. \qquad ...(27)$$

Similarly, using eqns. (3), (8), (9), (14), (17) and (18), one may obtain the transformations connecting the solutions in $(z_\psi; y_\psi)$-, $(z_P; y_P)$- and $(z_\rho; y_\rho)$-planes. From view point of astrophysical applications, more interesting case is the invesigation of "approximate" analytical solution of one of the above generalized first-order differential equations, say, eqn. (5) (as the necessity of using first-order equations to astrophysical problems is badly felt when one faces the difficulty of itegration of isothermal equation (1) by known methods) which is given in section 4.

4. Approximate Analytical Solutions of the Generalized First-Order Equation (5)

Let us assume a series expansion of the form

$$u_\psi = (m + 1) + av_\psi + bv_\psi^2 + cv_\psi^3 + dv_\psi^4 + ... \qquad ...(28)$$

which satisfies the conditions $u_\psi \to m + 1$, $v_\psi \to 0$ as $\xi \to 0$. We make use of this series in isothermal equation (5) and determine the coefficients $a, b, c, d, ...,$ successively by equating the coefficients of like powers of v_ψ. Thus, the series, including terms up to v_ψ^4, is found to be

$$u_\psi = (m+1) - \frac{1+m}{3+m} v_\psi - \frac{m^2-1}{(3+m)^2(5+m)} v_\psi^2$$
$$- \frac{(m^2-1)(4m+2)}{(3+m)^3(7+m)(5+m)} v_\psi^3$$
$$- \frac{(m^2-1)(23m^3+143m^2+113m+9)}{(3+m)^4(5+m)^2(9+m)(7+m)} v_\psi^4 . \qquad \text{...(29)}$$

For $m = 1$ (cylindrical cases), the foregoing series is expressed in simple form

$$u_\psi = 2 - \tfrac{2}{3} v_\psi. \qquad \text{...(30)}$$

The nature of the solution in (u_ψ; v_ψ)-plane is obvious. Therefore we need only to discuss the approximate analytical solutions following the method of Padé (2, 2) approximants* [which is previously applied by Seidov and Sharma (1979), Seidov *et al.* (1979a, b) to the cases of polytropic and degenerate configurations] for $m = 2$ and $m = 0$.

Case 1 : $m = 2$ — In this case, series in (29) can be written as

$$u_\psi = 3 - \frac{3}{5} v_\psi - \frac{3}{(5)^2\, 7} v_\psi^2 - \frac{2}{(5)^2\, 21} v_\psi^3 - \frac{991}{(5)^4\,(7)^2\, 33} v_\psi^4 . \qquad \text{...(31)}$$

Now, we may express the function u_ψ as Padé (2, 2) approximant:

$$u_\psi = 3. \frac{1 + A v_\psi + B v_\psi^2}{1 + C v_\psi + D v_\psi^2}. \qquad \text{...(32)}$$

On evaluating the coefficients A, B, C, and D from eqns. (31) and (32), we obtain u_ψ, finally, in the form

$$u_\psi = 3. \frac{22193325 - 10266480 v_\psi + 1064343 v_\psi^2}{7397775 - 1942605 v_\psi + 8533 v_\psi^2}. \qquad \text{...(33)}$$

Case 2 : $m = 0$ — From the generalized series expansion in (29), we find, in this case,

$$u_\psi = 1 - \frac{1}{3} v_\psi + \frac{1}{(3)^2\, 5} v_\psi^2 + \frac{2}{(3)^3\, 35} v_\psi^3 + \frac{1}{(3)^4\, 175} v_\psi^4 . \qquad \text{...(34)}$$

*See Appendix

Proceeding exactly in a similar way as in Case 1, Padé (2, 2) approximant is given by

$$u_\psi = \frac{5355 - 2100 v_\psi + 237 v_\psi^2}{5355 - 315 v_\psi + 13 v_\psi^2}. \qquad \text{...(35)}$$

Table I given below shows run of u_ψ for $m = 0$ and 2; upper and lower entries in u_ψ representing values calculated from our approximate and series solutions, respectively [eqns. (33), (31); (35); (34)]. In view of our previous studies (Seidov and Sharma 1979, Seidov *et al.* 1979a, b) that the approximate analytical solutions are very close to the numerical solutions; Table I brings out clearly the region of the validity of the series solutions.

TABLE I

Properties of isothermal spherical and plane-symmetric configurations

	$m = 2$	$m = 0$
v	u_ψ	u_ψ
0.0	3.0000000	1.0000000
	3.0000000	1.0000000
0.2	2.8792821	0.9342392
	2.8792823	0.9342393
0.4	2.7569853	0.8703594
	2.7569883	0.8703595
0.6	2.6328552	0.8084661
	2.6328788	0.8084663
0.8	2.5065717	0.7486676
	2.5066766	0.7486680
1.0	2.3777256	0.6910746
	2.3780672	0.6910758
1.4	2.1100479	0.5829601
	2.1121804	0.5829671
1 8	1.8230130	0.4850562
	1.8319476	0.4850827
2.2	1.5033087	0.3983305
	1.5334968	0.3984092
2.6	1.1209098	0.3237800
	1.2123531	0.3239748
3.0	0.5908410	0.2491776
	0.8634378	0.2628532

Physical Significance of the Generalized First-order Differential Equation (5)

We have seen above that the generalized first-order differential equation between u_ψ and v_ψ is given by

$$u_\psi = m - 1 \qquad \text{...(36)}$$

which is a line parallel to the v_ψ axis. The locus of points at which the curves have vertical tangents is given by

$$u_\psi + v_\psi = m + 1. \qquad \text{...(37)}$$

The point of intersection of the two loci (36) and (37) are

$$v_\psi = 2; \; u_\psi = m - 1. \qquad \text{...(38)}$$

In view of the homology theorem that any other member of the family $\{\psi(\xi)\}$ will lead to the same $(u_\psi; v_\psi)$-curve, therefore, to consider the *E*-solutions near the origin $(\xi_\psi \sim 0)$, we shall study, in brief, the behaviours of u_ψ and v_ψ at $\xi_\psi \sim 0$:

The generalized series expansion $\psi(\xi_\psi)$, obtainable from (1), is given by

$$\psi(\xi_\psi) = \frac{1}{2(1+m)}\xi_\psi^2 - \frac{1}{2^3(3+m)(1+m)}\xi_\psi^4 + \frac{2+m}{2^3 3(1+m)^2(5+m)(3+m)}\xi_\psi^6 - \frac{3m^2+16m+17}{2^6 3(1+m)^3(7+m)(5+m)(3+m)}\xi_\psi^8 + \frac{6m^4+79m^3+354m^2+629\,m+372}{2^6 15(1+m)^4(3+m)^2(9+m)(7+m)(5+m)}\xi_\psi^{10}. \qquad \text{...(39)}$$

Therefore we have

$$u_\psi = \frac{\xi_\psi e^{-\psi}}{\psi'} \sim (m+1)\left[1 - \frac{1}{(3+m)(1+m)}\xi_\psi^2\right] \quad (\xi_\psi \to 0) \qquad \text{...(40)}$$

$$v_\psi = \xi_\psi \psi' \sim \frac{1}{1+m}\xi_\psi^2 \qquad (\xi_\psi \to 0). \qquad \text{...(41)}$$

Thus, the *E*-curve passes through the point

$$u_\psi = m + 1, \quad v_\psi = 0 \qquad (\xi_\psi = 0) \qquad \text{...(42)}$$

that is, $(u_\psi; v_\psi)$ variables have the values $(1 + m, 0)$ at the centre $\xi_\psi = 0$; at this point $dv/du_\psi = -(3+m)/(1+m)$. Further, with plane-symmetric, cylindrical and spherical polytropes of $m = 0$, 1 and 2, we have $u_\psi(\xi_\psi) = u_\psi(-\xi_\psi)$; $v_\psi(\xi_\psi) = v(-\xi_\psi)$, therefore, the solutions of astrophysical interest lie in the positive quadrant $u_\psi \geqslant 0$, $v_\psi \geqslant 0$. We can express the $(u_\psi; v_\psi)$ variables in the form

$$u_\psi = \frac{\rho(\xi_\psi)}{\bar{\rho}(\xi_\psi)}; \quad v_\psi = \frac{\xi_\psi M(\xi_\psi)}{2\lambda\alpha} \qquad (m = 0) \qquad \ldots(43)$$

$$u_\psi = \frac{3\rho(\xi_\psi)}{\bar{\rho}(\xi_\psi)}; \quad v_\psi = \frac{1}{4\pi\lambda\alpha^3}\,\frac{M(\xi_\psi)}{\xi_\psi} \qquad (m = 2) \qquad \ldots(44)$$

($M(\xi_\psi)$ = mass). Equations (43) and (44) explain clearly the physical significance of Table I.

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APPENDIX

From eqns. (31) and (32) values of the coefficients A, B, C, and D, on equating like powers in v, are found to be

$$\left.\begin{aligned} A &= -\frac{1552}{3355}, \quad B = \frac{354781}{7397775}, \\ C &= -\frac{881}{3355}, \quad D = \frac{8533}{7397775}. \end{aligned}\right\} \text{(spherical case)} \qquad \ldots\text{(I)}$$

Similarly, using the series (34) and the assumed expression for Padé (2, 2) approximant

$$u_\psi = \frac{1 + A'v_\psi + B'v_\psi^2}{1 + C'v_\psi + D'v_\psi^2} \qquad \ldots\text{(II)}$$

we have

$$\left.\begin{aligned} A' &= -\frac{20}{51}, \quad B' = \frac{237}{5355}, \\ C' &= -\frac{1}{17}, \quad D' = \frac{13}{5355}. \end{aligned}\right\} \text{(plane-symmetric case)} \qquad \ldots\text{(III)}$$

Substitution of (III) in (II) yields the relation (35).

Thus we find that the technique of Padé approximants (Baker 1974) provides a good way for obtaining analytical solutions of both linear and non-linear differential equations, with prescribed boundary conditions. One may, however, expect a better accuracy [in 6th or even higher places of decimal] in the results by considering higer-order Padé approximants, for example, Padé (3, 3) approximant in which case calculations might appear inconveniently lengthy. However, from view point of astrophysical applications Padé (2, 2) approximant looks convenient and safe*. As concluded in our previous works (Seidov and Sharma 1979, Seidov *et al.* 1979a, b), the main advantages of Padé approximants are: (i) the solutions are far more exact than the series solutions and are very close to the numerical solutions and (ii) the boundary values of physical parameters (as in case of polytropic and degenerate configurations) are easily determinable even without use of computers, programmes and Runge-Kutta method (Kunz 1957)] (that is, calculations can be perfomed with the help of an electronic pocket calculator). Thus, the present method is economical too.

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