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DISTRIBUTION OF STRESSES AROUND A PENNY SHAPED CRACK IN A TRANSVERSELY ISOTROPIC SEMI-INFINITE MEDIUM

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An axi-symmetric problem of a semi-infinite transversely isotropic medium containing a penny shaped crack is considered. A general solution in terms of two potential functions is presented. Here the free-edge is assumed to be stress free. The mixed boundary conditions lead to two Fredholm integral equations of the second kind, which are solved by the use of Fox and Goodwin (1953) method. Numerical solutions for some practical materials like cadmium, magnesium are carried out and the percentage of increase in stress intensity factor due to presence of boundary is discussed.

1. INTRODUCTION

Nowadays the interest in fracture mechanics has focussed the attention of many research workers to the problem of determining stress in the vicinity of cracks. A comprehensive survey of crack problems in mathematical theory of elasticity has been given by Sneddon and Lowengrub (1969). Though considerable progress has been made in isotropic materials, problems of distribution of stresses around cracks in anisotropic medium are few in number. This may be due, in part, to mathematical complexity of such problems and also to the fact that engineering structures have generally been fabricated from materials which are essentially isotropic. But in recent years with the increased usage of anisotropic materials in engineering applications, the interest in anisotropic elasticity has grown considerably.

In this paper following Parhi and Atsumi (1975) we have expressed the displacements and stresses in a transversely isotropic medium in terms of two potential functions. The solution is then obtained for a penny-shaped crack of radius a situated at a distance h from the free boundary and is opened by the application of internal pressure.

The free boundary is assumed to be stress free. The corresponding isotropic problem has been solved by Srivastava and Singh (1969). Because of the use of two harmonic functions the calculations are found to be easier than those of isotropic case, where a bi-harmonic function is used.

In this paper numerical calculations for some practical materials like magnesium and cadmium are reported when the crack is opened by the application of constant internal normal pressure.

The percentage of increase of normal stress intensity factor due to finite boundary and the ratio of the shear stress intensity factor for the semi-infinite medium with that of normal stress intensity factor for the infinite medium have been calculated. These results along with those of the isotropic case (Srivastava and Singh 1969) are presented graphically (Figs. 4 and 5).

2. Formulation of the Problem

Here we take the displacement vector as $(u, 0, w)$ and components of the stress tensor as σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$ and σ_{rz}. Let us assume that the body is divided into two domains (Fig. 1):

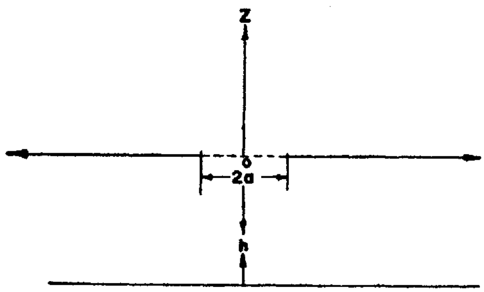

Fig. 1. Geometry of the problem.

(1) The layer $-h \leqslant z < 0$, where h is the distance of the crack from the free boundary; and

(2) the half space $0 \leqslant z < \infty$.

The boundary conditions can be written as

$$\sigma_{zz}(r, -h) = \sigma_{rz}(r, -h) = 0 \quad \forall\, r \qquad \text{...(2.1)}$$

$$\left.\begin{array}{l} \sigma_{zz}(r, 0^-) = \sigma_1(r),\ \sigma_{zz}(r, 0^+) = \sigma_2(r) \\ \sigma_{rz}(r, 0^-) = \tau_1(r),\ \sigma_{rz}(r, 0^+) = \tau_2(r) \end{array}\right\} \text{ for } |\, r \,| < a. \qquad \text{...(2.2)}$$

The continuity of displacements and stresses across $z = 0$ outside the crack requires the following additional conditions:

$$\left.\begin{array}{l} u(r, 0^+) = u(r, 0^-),\ \sigma_{zz}(r, 0^+) = \sigma_{zz}(r, 0^-) \\ w(r, 0^+) = w(r, 0^-),\ \sigma_{rz}(r, 0^+) = \sigma_{rz}(r, 0^-) \end{array}\right\} \text{ for } 0 \leqslant r < \infty. \qquad \text{...(2.3)}$$

3. Basic Equations

We take as usual the cylindrical coordinate system (r, θ, z), where z-axis is parallel to the axis of symmetry of the material.

We introduce a set of dimensionless parameters δ_i, and λ_i ($i = 1, 2$) which are dependent upon elastic constants c_{ij}, where δ_1, δ_2 are roots of the quadratic equation

$$c_{11}c_{44}\delta^4 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}]\,\delta^2 + c_{33}c_{44} = 0 \qquad \ldots(3.1)$$

and λ_1, λ_2 are given by

$$\lambda_i = \frac{c_{11}\delta_i^2 - c_{44}}{c_{13} + c_{44}}, \quad i = 1, 2. \qquad \ldots(3.2)$$

A set of displacements and stresses which satisfy equilibrium equations can be derived from two harmonic functions $\varphi_1(r, z_1)$ and $\varphi_2(r, z_2)$, where $\varphi_1(r, z_1)$ is harmonic in (r, θ, z_1) space and $\varphi_2(r, z_2)$ is harmonic in (r, θ, z_2) space and $z_i = z/\delta_i$ ($i = 1, 2$).

These displacements and stresses are

$$\left.\begin{aligned} u &= \frac{\partial}{\partial r}[\varphi_1(r, z_1) + \varphi_2(r, z_2)] \\ w &= \frac{\lambda_1}{\delta_1}\frac{\partial}{\partial z_1}\varphi_1(r, z_1) + \frac{\lambda_2}{\delta_2}\frac{\partial\varphi_2(r, z_2)}{\partial z_2} \end{aligned}\right\} \qquad \ldots(3.3)$$

$$\left.\begin{aligned} \frac{\sigma_{zz}}{c_{44}} &= (1 + \lambda_1)\frac{\partial^2\varphi_1(r, z_1)}{\partial z_1^2} + (1 + \lambda_2)\frac{\partial^2\varphi_2(r, z_2)}{\partial z_2^2} \\ \frac{\sigma_{rz}}{c_{44}} &= \frac{1 + \lambda_1}{\delta_1}\frac{\partial^2\varphi_1(r, z_1)}{\partial r\partial z_1} + \frac{1 + \lambda_2}{\delta_2}\frac{\partial^2\varphi_2(r, z_2)}{\partial r\partial z_2} \\ -\frac{\sigma_{rr}}{c_{44}} &= \frac{1 + \lambda_1}{\delta_1^2}\frac{\partial^2\varphi_1(r, z_1)}{\partial z_1^2} + \frac{1 + \lambda_2}{\delta_2^2}\frac{\partial^2\varphi_2(r, z_2)}{\partial z_2^2} \\ &\quad + \frac{c_{11} - c_{12}}{c_{44}}\frac{1}{r}\frac{\partial}{\partial r}[\varphi_1(r, z_1) + \varphi_2(r, z_2)] \\ -\frac{\sigma_{\theta\theta}}{c_{44}} &= \frac{1 + \lambda_1}{\delta_1^2}\frac{\partial^2\varphi_1(r, z_1)}{\partial z_1^2} + \frac{1 + \lambda_2}{\delta_2^2}\frac{\partial^2\varphi_2(r, z_2)}{\partial z_2^2} \\ &\quad + \frac{c_{11} - c_{12}}{c_{44}}\frac{\partial^2}{\partial r^2}[\varphi_1(r, z_1) + \varphi_2(r, z_2)]. \end{aligned}\right\} \qquad \ldots(3.4)$$

4. Solution for Semi-infinite Region

To solve the mixed boundary value problem we take the two potential functions defined in section 3 for semi-infinite region as follows:

$$\varphi_i(r, z_i) = \int_0^\infty \xi^{-1}A_i(\xi)\, e^{-\xi z_i}J_0(\xi r)\, d\xi \quad (i = 1, 2). \qquad \ldots(4.1)$$

Then using (4.1) the expression for components of stress tensor and displacement vector can be derived from (3.3) and (3.4) as

$$\left.\begin{aligned} u &= -\sum_{i=1}^{2} \int_0^\infty A_i(\xi)\, e^{-\xi z_i} J_1(\xi r)\, d\xi \\ w &= -\sum_{i=1}^{2} \frac{\lambda_i}{\delta_i} \int_0^\infty A_i(\xi)\, e^{-\xi z_i} J_0(\xi r)\, d\xi \end{aligned}\right\} \qquad \ldots(4.2)$$

(a) $$\frac{\sigma_{zz}}{c_{44}} = \sum_{i=1}^{2} (1+\lambda_i) \int_0^\infty \xi A_i(\xi)\, e^{-\xi z_i} J_0(\xi r)\, d\xi,$$

(b) $$\frac{\sigma_{rz}}{c_{44}} = \sum_{i=1}^{2} \frac{(1+\lambda_i)}{\delta_i} \int_0^\infty \xi A_i(\xi)\, e^{-\xi z_i} J_1(\xi r)\, d\xi,$$

(c) $$-\frac{\sigma_{rr}}{c_{44}} = \sum_{i=1}^{2} \left[\frac{1+\lambda_i}{\delta_i^2} \int_0^\infty \xi A_i(\xi)\, e^{-\xi z_i} J_0(\xi r)\, d\xi - \frac{c_{11}-c_{12}}{c_{44}} \frac{1}{r} \int_0^\infty A_i(\xi)\, e^{-\xi z_i} J_1(\xi r)\, d\xi \right],$$

(d) $$-\frac{\sigma_{\theta\theta}}{c_{44}} = \sum_{i=1}^{2} \left[\left\{ \frac{1+\lambda_i}{\delta_i^2} - \frac{c_{11}-c_{12}}{c_{44}} \right\} \int_0^\infty \xi A_i(\xi)\, e^{-\xi z_i} J_0(\xi r)\, d\xi + \frac{c_{11}-c_{12}}{c_{44}} \frac{1}{r} \int_0^\infty A_i(\xi)\, e^{-\xi z_i} J_1(\xi r)\, d\xi \right]. \qquad \ldots(4.3)$$

5. Solution for the Region $-h \leqslant z \leqslant 0$

In this case the potential functions $\varphi_i(r, z_i)$, $(i = 1, 2)$ can be taken as

$$\varphi_i = \int_0^\infty \xi^{-1} \{B_i(\xi) \sin \xi(z_i + h_i) + c_i(\xi) \cosh \xi(z_i + h_i)\}\, J_0(\xi r)\, d\xi,\ i = 1, 2. \qquad \ldots(5.1)$$

Utilizing (5.1) in (3.3) and (3.4) the expressions for the components of stress tensor and displacement vector become

$$u = -\sum_{i=1}^{2} \int_0^\infty \{B_i(\xi) \sinh \xi(z_i + h_i) + c_i(\xi) \cosh \xi(z_i + h_i)\}\, J_1(\xi r)\, d\xi) \qquad \ldots(5.2a)$$

$$w = \sum_{i=1}^{2} \left[\frac{\lambda_i}{\delta_i} \int_0^\infty \{B_i(\xi) \cosh \xi(z_i + h_i) + c_i(\xi) \sinh \xi(z_i + h_i)\} J_0(\xi r)\, d\xi \right] \quad \ldots(5.2b)$$

$$\left.\begin{aligned} \frac{\sigma_{zz}}{c_{44}} &= \sum_{i=1}^{2} \left[(1 + \lambda_i) \int_0^\infty \xi \{B_i(\xi) \sinh \xi(z_i + h_i) + c_i(\xi) \times \cosh \xi(z_i + h_i)\} J_0(\xi r)\, d\xi \right] \\ \frac{\sigma_{rz}}{c_{44}} &= - \sum_{i=1}^{2} \left[\frac{(1 + \lambda_i)}{\delta_i} \int_0^\infty \xi \{B_i(\xi) \cosh \xi(z_i + h_i) + c_i(\xi) \times \sinh \xi(z_i + h_i)\} J_1(\xi r)\, d\xi \right]. \end{aligned}\right\} \quad \ldots(5.3)$$

Since the stress components σ_{rr}, $\sigma_{\theta\theta}$ are nowhere utilised in our problem, we do not mention them here.

Using boundary condition (2.1) we get

$$\left.\begin{aligned} c_1(\xi) &= - \frac{(1 + \lambda_2)}{(1 + \lambda_1)} c_2(\xi) \\ B_1(\xi) &= - \frac{(1 + \lambda_2)}{(1 + \lambda_1)} \frac{\delta_1}{\delta_2} B_2(\xi). \end{aligned}\right\} \quad \ldots(5.4)$$

Substituting (5.4) in (5.2) and (5.3) and dropping the subscript '2' from c_2 and B_2 we get

$$u = - \int_0^\infty \left[\left\{ - \frac{(1 + \lambda_2)}{(1 + \lambda_1)} \frac{\delta_1}{\delta_2} \sinh \xi(z_1 + h_1) + \sinh \xi(z_2 + h_2) \right\} B(\xi) + \left\{ - \frac{(1 + \lambda_2)}{(1 + \lambda_1)} \cosh \xi(z_1 + h_1) + \cosh \xi(z_2 + h_2) \right\} c(\xi) \right] J_1(\xi r)\, d\xi \quad \ldots(5.5)$$

$$w = \int_0^\infty \left[\left\{ - \frac{\lambda_1(1 + \lambda_2)}{\delta_2(1 + \lambda_1)} \cosh \xi(z_1 + h_1) + \frac{\lambda_2}{\delta_2} \cosh \xi(z_2 + h_2) \right\} B(\xi) + \left\{ - \frac{\lambda_1(1 + \lambda_2)}{\delta_1(1 + \lambda_1)} \sinh \xi\, (z_1 + h_1) + \frac{\lambda_2}{\delta_2} \sinh \xi(z_2 + h_2) \right\} c(\xi) \right] \times J_0(\xi r)\, d\xi \quad \ldots(5.6)$$

$$\frac{\sigma_{zz}}{c_{44}} = \int_0^\infty \xi \Big[\Big\{ -\frac{\delta_1(1+\lambda_2)}{\delta_2} \sinh \xi(z_1+h_1) + (1+\lambda_2) \sinh \xi(z_2+h_2) \Big\}$$
$$\times B(\xi) + \{-(1+\lambda_2) \cosh \xi(z_1+h_1) + (1+\lambda_2) \cosh \xi(z_2+h_2)\}$$
$$\times c(\xi) \Big] J_0(\xi r)\, d\xi \qquad \ldots(5.7)$$

$$\frac{\sigma_{rz}}{c_{44}} = -\int_0^\infty \xi \Big[\Big\{ -\frac{(1+\lambda_2)}{\delta_2} \cosh \xi(z_1+h_1) + \frac{(1+\lambda_2)}{\delta_2}$$
$$\times \cosh \xi(z_2+h_2) \Big\} B(\xi) + \Big\{ -\frac{(1+\lambda_2)}{\delta_1} \sinh \xi(z_1+h_1)$$
$$+ \frac{(1+\lambda_2)}{\delta_2} \sinh \xi(z_2+h_2) \Big\} c(\xi) \Big] J_1(\xi r)\, d\xi. \qquad \ldots(5.8)$$

6. Reduction of the Problem to a System of Simultaneous Dual Integral Equations

Making use of the boundary conditions (2.2) and assuming that

$$\tau_1(r) = \tau_2(r) = -t_0 t(r/a) \quad \text{and} \quad \sigma_1(r) = \sigma_2(r) = -p_0 p(r/a)$$

we get

$$\int_0^\infty \xi \Big[(1+\lambda_2) \Big\{ -\frac{\delta_1}{\delta_2} \sinh \xi h_1 + \sinh \xi h_2 \Big\} B(\xi)$$
$$+ (1+\lambda_2) \{-\cosh \xi h_1 + \cosh \xi h_2\} c(\xi) \Big] J_0(\xi r)\, d\xi$$
$$= \sum_{i=1}^{2} \int_0^\infty \xi (1+\lambda_i) A_i(\xi) J_0(\xi r)\, d\xi = -\frac{p_0 p(r/a)}{c_{44}} \qquad \ldots(6.1)$$

$$(1+\lambda_2) \int_0^\infty \xi \Big[\delta_2^{-1} \{-\cosh \xi h_1 + \cosh \xi h_2\} B(\xi)$$
$$+ \Big\{ \sum_{i=1}^{2} (-1)^i \frac{\sinh \xi h_i}{\delta_i} \Big\} c(\xi) \Big] J_1(\xi r)\, d\xi$$
$$= \sum_{i=1}^{2} \int_0^\infty \xi \frac{(1+\lambda_i)}{\delta_i} A_i(\xi) J_1(\xi r)\, d\xi = -\frac{t_0 t(r/a)}{c_{44}}. \qquad \ldots(6.2)$$

Again from the continuity conditions (2.3) we get

$$\int_0^\infty \Big[\Big\{-\frac{(1+\lambda_2)}{(1+\lambda_1)}\frac{\delta_1}{\delta_2}\sinh\xi h_1+\sinh\xi h_2\Big\}B(\xi)$$
$$+\Big\{-\frac{(1+\lambda_2)}{(1+\lambda_1)}\cosh\xi h_1+\cosh\xi h_2\Big\}c(\xi)-A_1(\xi)-A_2(\xi)\Big]$$
$$\times J_1(\xi r)\,d\xi=0 \qquad \ldots(6.3)$$

$$\int_0^\infty \Big[\Big\{-\frac{\lambda_1(1+\lambda_2)}{\delta_2(1+\lambda_1)}\cosh\xi h_1+\frac{\lambda_2}{\delta_2}\cosh\xi h_2\Big\}B(\xi)$$
$$+\Big\{-\frac{\lambda_1(1+\lambda_2)}{\delta_1(1+\lambda_1)}\sinh\xi h_1+\frac{\lambda_2}{\delta_2}\sinh\xi h_2\Big\}c(\xi)$$
$$+\frac{\lambda_1}{\delta_1}A_1(\xi)+\frac{\lambda_2}{\delta_2}A_2(\xi)\Big]J_0(\xi r)\,d\xi=0. \qquad \ldots(6.4)$$

From (6.1) and (6.2) we have

$$X(\xi)=(1+\lambda_2)\Big\{-\frac{\delta_1}{\delta_2}\sinh\xi h_1+\sinh\xi h_2\Big\}B(\xi)+(1+\lambda_2)$$
$$\times\{\sum_{i=1}^{2}(-1)^i\cosh\xi h_i\}\,c(\xi)=\sum_{i=1}^{2}(1+\lambda_i)\,A_i(\xi) \qquad \ldots(6.5)$$

$$Y(\xi)=(1+\lambda_2)\,\delta_2^{-1}\{\sum_{i=1}^{2}(-1)^i\cosh\xi h_i\}\,B(\xi)$$
$$+(1+\lambda_2)\Big\{\sum_{i=1}^{2}(-1)^i\frac{\sinh\xi h_i}{\delta_i}\Big\}c(\xi)=-\sum_{i=1}^{2}\frac{(1+\lambda_i)}{\delta_i}A_i(\xi). \qquad \ldots(6.6)$$

Again let us write

$$M(\xi)=\Big\{-\frac{(1+\lambda_2)}{(1+\lambda_1)}\frac{\delta_1}{\delta_2}\sinh\xi h_1+\sinh\xi h_2\Big\}B(\xi)$$
$$+\Big\{-\frac{\lambda_1(1+\lambda_2)}{\delta_1(1+\lambda_1)}\sinh\xi h_1+\cosh\xi h_2\Big\}c(\xi)-A_1(\xi)-A_2(\xi) \qquad \ldots(6.7)$$

$$N(\xi)=\Big\{-\frac{\lambda_1(1+\lambda_2)}{\delta_2(1+\lambda_1)}\cosh\xi h_1+\frac{\lambda_2}{\delta_2}\cosh\xi h_2\Big\}B(\xi)$$
$$+\Big\{-\frac{\lambda_1(1+\lambda_2)}{\delta_1(1+\lambda_1)}\sinh\xi h_1+\frac{\lambda_2}{\delta_2}\sinh\xi h_2\Big\}c(\xi)$$
$$+\frac{\lambda_1}{\delta_1}A_1(\xi)+\frac{\lambda_2}{\delta_2}A_2(\xi). \qquad \ldots(6.8)$$

From eqns. (6.5) – (6.8) we get

$$X(\xi) = PN(\xi) + I(z)\, N(\xi) + J(z)\, M(\xi) \quad \text{...(6.9)}$$

$$Y(\xi) = QM(\xi) + K(z)\, M(\xi) + L(z)\, N(\xi) \quad \text{...(6.10)}$$

where

$$\left.\begin{aligned} &\xi h = z \\ &I(z) = -\frac{P}{(\delta_1 - \delta_2)^2}\,[\delta_1(\delta_1 + \delta_2)\, e^{-2z/\delta_1} + \delta_2(\delta_1 + \delta_2)\, e^{-2z/\delta_2} \\ &\qquad - 4\delta_1\delta_2 e^{-(z/\delta_1)-(z/\delta_2)}] \end{aligned}\right\} \quad \text{...(6.11)}$$

$$\left.\begin{aligned} &J(z) = \frac{P(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2}\,[e^{-z/\delta_1} - e^{-z/\delta_2}]^2 \\ &K(z) = -\frac{Q}{(\delta_1 - \delta_2)^2}\,[\delta_2(\delta_1 + \delta_2)\, e^{-2z/\delta_1} \\ &\qquad + \delta_1(\delta_1 + \delta_2)\, e^{-2z/\delta_2} - 4\delta_1\delta_2 e^{-(z/\delta_1)-(z/\delta_2)}] \\ &L(z) = J(z) \\ &P = -\frac{(1 + \lambda_1)(1 + \lambda_2)(\delta_1 - \delta_2)}{2(\lambda_2 - \lambda_1)} \\ &Q = \frac{P}{\delta_1\delta_2}. \end{aligned}\right\} \quad \text{...(6.12)}$$

7. Some Useful Results

Let us list below some important results for our ready reference. All these are found in Bateman (1954):

$$\int_0^\infty e^{-pt} \sin \alpha t \sin \beta t \, dt = \frac{2\alpha\beta p}{[p^2 + (\alpha + \beta)^2]\,[p^2 + (\alpha - \beta)^2]} = Q_1^* (p, \alpha, \beta) \quad \text{...(7.1)}$$

$$\int_0^\infty e^{-pt} \cos \alpha t \sin \beta t \, dt = \frac{\beta(p^2 - \alpha^2 + \beta^2)}{[p^2 + (\alpha + \beta)^2]\,[p^2 + (\alpha - \beta)^2]} = Q_2^* (p, \alpha, \beta) \quad \text{...(7.2)}$$

$$\int_0^\infty e^{-pt} \cos \alpha t \cos \beta t \, dt = \frac{p(p^2 + \alpha^2 + \beta^2)}{[p^2 + (\alpha + \beta)^2]\,[p^2 + (\alpha - \beta)^2]} = Q_3^* (p, \alpha, \beta) \quad \text{...(7.3)}$$

$$\int_0^\infty t^{-1} e^{-pt} \sin \alpha t \sin \beta t \, dt = \tfrac{1}{4} \log \frac{p^2 + (\alpha + \beta)^2}{p^2 + (\alpha - \beta)^2} = Q_4^* (p, \alpha, \beta) \quad \text{...(7.4)}$$

$$\int t^{-1}e^{-pt}\sin\alpha t\cos\beta t\,dt = \tfrac{1}{2}\tan^{-1}\frac{2\alpha p}{p^2-\alpha^2+\beta^2} = Q_5^*(p,\alpha,\beta) \quad \ldots(7.5)$$

$$\int t^{-2}e^{-pt}\sin\alpha t\sin\beta t\,dt = \tfrac{1}{2}\alpha\tan^{-1}\frac{2\beta p}{p^2+\alpha^2-\beta^2} + \tfrac{1}{2}\beta\tan^{-1}\frac{2\alpha p}{p^2-\alpha^2+\beta^2} + \tfrac{1}{4}p\log\frac{p^2+(\alpha-\beta)^2}{p^2+(\alpha+\beta)^2} = Q_6^*(p,\alpha,\beta). \quad \ldots(7.6)$$

8. Solution of the Simultaneous Dual Integral Equations

Now eqns. (6.1) to (6.4) with the help of eqns. (6.7) to (6.10) reduce to a system of quadruple integral equations as follows:

$$\left.\begin{aligned}\int_0^\infty M(\xi)\,J_1(\xi r)\,d\xi &= 0\\ \int_0^\infty N(\xi)\,J_0(\xi r)\,d\xi &= 0\end{aligned}\right\}\ \text{for}\ a<r<\infty \quad \ldots(8.1)$$

$$\left.\begin{aligned}\int_0^\infty \xi\,[PN(\xi)+I(z)\,N(\xi)+J(z)\,M(\xi)]\,J_0(\xi r)\,d\xi &= \frac{p_0p(r/a)}{c_{44}}\\ \int_0^\infty \xi\,[QM(\xi)+K(z)\,M(\xi)+L(z)\,N(\xi)]\,J_1(\xi r)\,d\xi &= \frac{t_0t(r/a)}{c_{44}}\end{aligned}\right\}\ \text{for}\ 0<r<a. \quad \ldots(8.2)$$

As in Srivastava and Singh (1969), let the trial solutions be

$$\begin{aligned}M(\xi) &= \xi^{1/2}\int_0^a m(t)\,J_{3/2}(\xi t)\,dt\\ &= -\left(\frac{2}{\pi}\right)^{1/2}\int_0^a t^{1/2}m(t)\,\frac{d}{dt}\left\{\frac{\sin\xi t}{\xi t}\right\}dt\\ &= \frac{1}{\xi}\left(\frac{2}{\pi}\right)^{1/2}\left[-\frac{m(a)\sin\xi a}{a^{1/2}}+\int_0^a\frac{\sin\xi t}{t}\,\frac{d}{dt}\{t^{1/2}m(t)\}\,dt\right] \quad \ldots(8.3)\end{aligned}$$

$$N(\xi) = \int_0^a n(t)\sin\xi t\,dt = -\frac{\cos\xi a}{\xi}\,n(a)+\frac{1}{\xi}\int_0^a\frac{d}{dt}n(t)\cos\xi t\,dt. \quad \ldots(8.4)$$

Using dimensionless quantities

$$\xi h = Z,\ \frac{t}{a} = T,\ \frac{r}{a} = R,\ \frac{u}{a} = U,\ \frac{h}{a} = G,$$

$$\frac{m(t)\,c_{44}}{a^{3/2}p_0} = M'(T),\ \frac{n(t)\,c_{44}}{ap_0} = N'(T) \qquad \ldots(8.5)$$

the integral eqns. (8.1) and (8.2) with the help of (8.3) and (8.4) after some manipulations reduce to

$$\left.\begin{aligned} N'(T) &= -\frac{2}{\pi P}\int_0^T \frac{Rp(R)}{(T^2-R^2)^{1/2}}\,dR + \int_0^1 [N'(U)\,\bar{R}(T,U) \\ &\qquad + M'(U)\,\bar{S}(T,U)]\,dU \\ M'(T) &= \frac{t_0}{Qp_0}\left(\frac{2}{\pi T}\right)^{1/2}\int_0^T \frac{R^2 t(R)}{(T^2-R^2)^{1/2}}\,dR + \int_0^1 [M'(U) \\ &\qquad \times \bar{R}_1(T,U) + N'(U)\,\bar{S}_1(T,U)]\,dU \end{aligned}\right\} \qquad \ldots(8.6)$$

and

$$\left.\begin{aligned} \bar{R}(T,U) &= -\frac{2}{\pi PG}\int_0^\infty I(z)\sin\left(\frac{zU}{G}\right)\sin\left(\frac{zT}{G}\right)dz \\ \bar{S}(T,U) &= -\frac{2}{\pi PG^{3/2}}\int_0^\infty z^{1/2}J(z)\,J_{3/2}\left(\frac{zU}{G}\right)\sin\left(\frac{zT}{G}\right)dz \\ \bar{R}_1(T,U) &= -\frac{T}{G^2Q}\int_0^\infty zK(z)\,J_{3/2}\left(\frac{zU}{G}\right)J_{3/2}\left(\frac{zT}{G}\right)dz \\ \bar{S}_1(T,U) &= -\frac{T}{G^{3/2}Q}\int_0^\infty z^{1/2}L(z)\sin\left(\frac{zU}{G}\right)J_{3/2}\left(\frac{zT}{G}\right)dz. \end{aligned}\right\} \qquad \ldots(8.7)$$

Then the kernels (8.7) are integrated in closed terms by the use of the results (7.1) to (7.6) and listed below:

$$\bar{R}(T,U) = \sum_{i=1}^{3} P_i,\ \bar{R}_1(T,U) = \sum_{i=1}^{12} R_i,\ \bar{S}(T,U) = \sum_{i=1}^{6} Q_i,\ \bar{S}_1(T,U) = \sum_{i=1}^{6} S_i,$$

$$P_1 = \frac{2\delta_1(\delta_1+\delta_2)}{\pi G(\delta_1-\delta_2)^2}\,Q_1^*\left(\frac{2}{\delta_1}, \frac{U}{G}, \frac{T}{G}\right),$$

[*equations* (8.8) *continued on p.* 139]

$$P_2 = \frac{2\delta_2(\delta_1 + \delta_2)}{\pi G(\delta_1 - \delta_2)} Q_1^* \left(\frac{2}{\delta_2}, \frac{U}{G}, \frac{T}{G}\right)$$

$$P_3 = -\frac{8\delta_1\delta_2}{\pi G(\delta_1 - \delta_2)^2} Q_1^* \left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{U}{G}, \frac{T}{G}\right)$$

$$Q_1 = -\left(\frac{2}{\pi U}\right)^{3/2} \frac{(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2} Q_4^* \left(\frac{2}{\delta_1}, \frac{U}{G}, \frac{T}{G}\right)$$

$$Q_2 = -\left(\frac{2}{\pi U}\right)^{3/2} \frac{(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2} Q_4^* \left(\frac{2}{\delta_2}, \frac{U}{G}, \frac{T}{G}\right)$$

$$Q_3 = \left(\frac{2}{\pi U}\right)^{3/2} \frac{(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2} 2Q_4^* \left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{U}{G}, \frac{T}{G}\right)$$

$$Q_4 = \left(\frac{2}{\pi}\right)^{3/2} \frac{(\delta_1 + \delta_2)}{U^{1/2}G(\delta_1 - \delta_2)^2} Q_2^* \left(\frac{2}{\delta_1}, \frac{U}{G}, \frac{T}{G}\right)$$

$$Q_5 = \left(\frac{2}{\pi}\right)^{3/2} \frac{(\delta_1 + \delta_2)}{U^{1/2}G(\delta_1 - \delta_2)^2} Q_2^* \left(\frac{2}{\delta_2}, \frac{U}{G}, \frac{T}{G}\right)$$

$$Q_6 = -\left(\frac{2}{\pi}\right)^{3/2} \frac{2(\delta_1 + \delta_2)}{U^{1/2}(\delta_1 - \delta_2)^2 G} Q_2^* \left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_1 = \frac{2TG\delta_2(\delta_1 + \delta_2)}{\pi(TU)^{2/3}(\delta_1 - \delta_2)^2} Q_6^* \left(\frac{2}{\delta_1}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_2 = \frac{2TG\delta_1(\delta_1 + \delta_2)}{\pi(TU)^{3/2}(\delta_1 - \delta_2)^2} Q_6^* \left(\frac{2}{\delta_2}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_3 = -\frac{8TG\delta_1\delta_2}{\pi(TU)^{3/2}(\delta_1 - \delta_2)^2} Q_6^* \left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_4 = -\frac{2\delta_2(\delta_1 + \delta_2)}{\pi(TU)^{1/2}(\delta_1 - \delta_2)^2} Q_5^* \left(\frac{2}{\delta_1}, \frac{T}{G}, \frac{U}{G}\right)$$

$$R_5 = -\frac{2\delta_1(\delta_1 + \delta_2)}{\pi(TU)^{1/2}(\delta_1 - \delta_2)^2} Q_5^* \left(\frac{2}{\delta_2}, \frac{T}{G}, \frac{U}{G}\right)$$

$$R_6 = \frac{8\delta_1\delta_2}{\pi(TU)^{1/2}(\delta_1 - \delta_2)^2} Q_5^* \left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{T}{G}, \frac{U}{G}\right)$$

$$R_7 = -\frac{2T^{1/2}\delta_2(\delta_1 + \delta_2)}{\pi U^{3/2}(\delta_1 - \delta_2)^2} Q_5^* \left(\frac{2}{\delta_1}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_8 = -\frac{2T^{1/2}\delta_1(\delta_1 + \delta_2)}{\pi U^{3/2}(\delta_1 - \delta_2)^2} Q_5^* \left(\frac{2}{\delta_2}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_9 = \frac{8\delta_1\delta_2 T^{1/2}}{\pi U^{3/2}(\delta_1 - \delta_2)^2} Q_5^* \left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{U}{G}, \frac{T}{G}\right)$$

[*equations* (8.8) *continued on p.* 140]

$$R_{10} = \frac{2T^{1/2}\delta_2(\delta_1 + \delta_2)}{\pi G U^{1/2}(\delta_1 - \delta_2)^2} Q_3^*\left(\frac{2}{\delta_1}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_{11} = \frac{2T^{1/2}\delta_1(\delta_1 + \delta_2)}{\pi G U^{1/2}(\delta_1 - \delta_2)^2} Q_3^*\left(\frac{2}{\delta_2}, \frac{U}{G}, \frac{T}{G}\right)$$

$$R_{12} = -\frac{8T^{1/2}\delta_1\delta_2}{\pi G(\delta_1 - \delta_2)^2 U^{1/2}} Q_3^*\left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{U}{G}, \frac{T}{G}\right)$$

$$S_1 = -\left(\frac{2}{\pi T}\right)^{1/2} \frac{P(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2} Q_4^*\left(\frac{2}{\delta_1}, \frac{T}{G}, \frac{U}{G}\right)$$

$$S_2 = -\left(\frac{2}{\pi T}\right)^{1/2} \frac{P(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2} Q_4^*\left(\frac{2}{\delta_2}, \frac{T}{G}, \frac{U}{G}\right)$$

$$S_3 = \left(\frac{2}{\pi T}\right)^{1/2} \frac{2P(\delta_1 + \delta_2)}{(\delta_1 - \delta_2)^2} Q_4^*\left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{T}{G}, \frac{U}{G}\right)$$

$$S_4 = \left(\frac{2T}{\pi}\right)^{1/2} \frac{P(\delta_1 + \delta_2)}{G(\delta_1 - \delta_2)^2} Q_2^*\left(\frac{2}{\delta_1}, \frac{T}{G}, \frac{U}{G}\right)$$

$$S_5 = \left(\frac{2T}{\pi}\right)^{1/2} \frac{P(\delta_1 + \delta_2)}{G(\delta_1 - \delta_2)^2} Q_2^*\left(\frac{2}{\delta_2}, \frac{T}{G}, \frac{U}{G}\right)$$

$$S_6 = -\left(\frac{2T}{\pi}\right)^{1/2} \frac{2P(\delta_1 + \delta_2)}{G(\delta_1 - \delta_2)^2} Q_2^*\left(\overline{\frac{1}{\delta_1} + \frac{1}{\delta_2}}, \frac{T}{G}, \frac{U}{G}\right). \quad \text{...(8.8)}$$

For constant pressure and vanishing shear we take $p(x) = 1$, $t(x) = 0$.

9. Stress Intensity Factors

Once the solutions of the integral eqns. (8.6) are obtained, stress intensity factors are calculated as follows

$$\left.\begin{aligned} K_1 &= \lim_{R\to 1} (R-1)^{1/2}\{\bar{\sigma}_{zz}(R, 0)\} = -\frac{PN'(1)}{\sqrt{2}} \\ K_2 &= \lim_{R\to 1} (R-1)^{1/2}\{\bar{\sigma}_{rz}(R, 0)\} = \frac{QM'(1)}{\sqrt{\pi}} \end{aligned}\right\} \quad \text{...(9.1)}$$

where

$$\left.\begin{aligned} \bar{\sigma}_{zz}(R, 0) &= \sigma_{zz}(r, 0)/c_{44} \\ \bar{\sigma}_{rz}(R, 0) &= \sigma_{rz}(r, 0)/c_{44} \end{aligned}\right\} \quad \text{...(9.2)}$$

are dimensionless normal and shearing stress components respectively.

For infinite medium $G \to \infty$, eqns. (8.6) reduce to

$$N'(T) = -\frac{2T}{\pi P}, \quad M'(T) = 0. \quad \text{...(9.3)}$$

The only non-zero stress intensity factor K_1 becomes the stress intensity factor for infinite medium which is denoted by K_∞ and it is due to the formula (9.1)

$$K_\infty = \frac{\sqrt{2}}{\pi}. \qquad \text{...(9.4)}$$

The percentage of increase of the stress intensity factor due to the presence of boundary (in semi-infinite medium) with regard to the infinite medium is calculated by the formula

$$\rho_1 = (K_1 - K_\infty) \times 100/K_\infty. \qquad \text{...(9.5)}$$

10. Numerical Results

Numerical results were obtained for constant pressure and zero shear. The computations were carried out in the I.B.M. 370/155 Computer at Computer Centre, I.I.T., Madras.

The Fredholm integral eqns. (8.6) are numerically solved by Fox and Goodwin (1953) method. The computations are carried out for values of $G = 1.05$, 1.1, 1.25, 1.4, 1.6667, 2.5, 3.0 and 4.0. For each value of G, the four kernel functions given by (8.8) are computed at the pivotal points of U and T which are 0, 0.1, 0.2, ..., 1.0.

Thus corresponding to every value of G, each of the integral equations of (8.6) is replaced by a system of eleven simultaneous linear equations. Hence in all we obtain a set of twenty-two linear simultaneous equations giving the values of twenty-two unknown function $N'(T)$ and $M'(T)$ at $T = 0.0, 0.1, 0.2, \ldots, 1.0$. It is observed from these equations that $N'(0) = M'(0) = 0$, hence the equations are reduced to only twenty equations and then are solved.

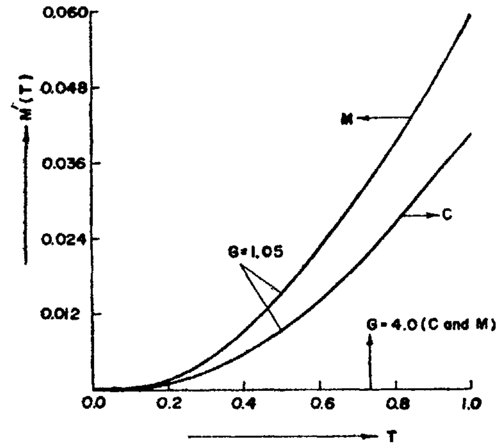

Fig. 2. Variation with T of $M'(T)$.

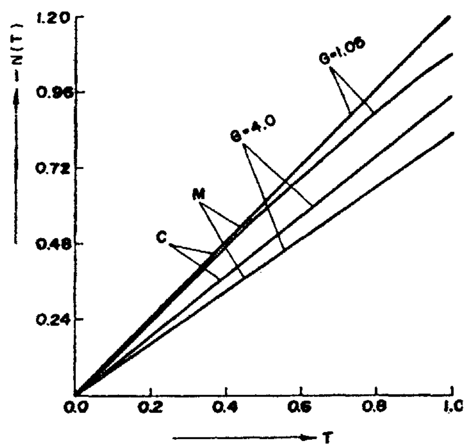

Fig. 3. Variation with T of $-N'(T)$.

(In the Figures C, M and I stand for cadmium, magnesium and isotropic respectively.)

The variation of $N'(T)$ and $M'(T)$ ($G = 1.05$ and 4.0) with T for cadmium and magnesium are shown in Figs. 2 and 3. It is observed that the curves for $M'(T)$ ($G = 1.05$) differ slightly from straight lines. Also when $G = 4.0$, the curve coincides with the axis of T.

The percentage of increase of normal stress intensity factor ρ_1 and the ratio $\tau_1 = K_2/K_\infty$ are calculated for cadmium and magnesium and are compared with the isotropic results of Srivastava and Singh (1969) and are presented in Figs. 4 and 5. The material constants are taken from (Huntington 1958) and mentioned in Table I.

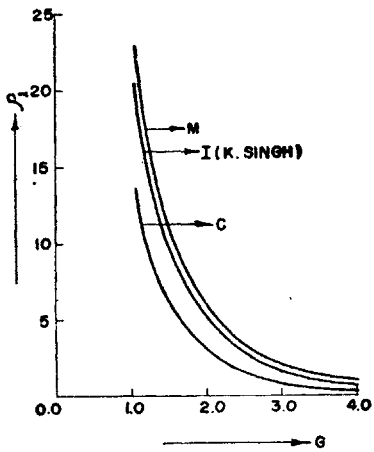

FIG. 4. Variation with G of ρ_1, percentage of increase in the normal stress intensity factor due to effect of finite distance of the crack from the free boundary.

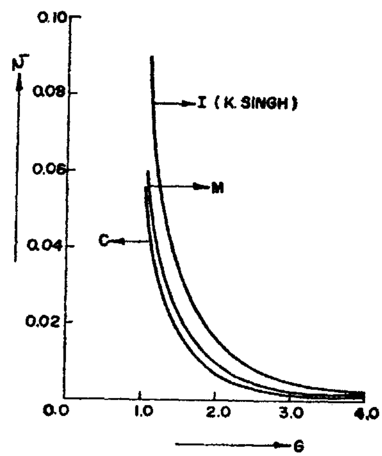

FIG. 5. The variation with G of $\tau_1 = K_2/K_\infty$.

TABLE I

Values of physical constants c_{ij} (in the units of 10^{10} N/m^2).

	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Cadmium	11.0	4.04	3.83	4.69	1.56
Magnesium	5.97	2.62	2.17	6.17	1.64

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