Indian J. pure appl. Math., 12(1) : 144–150, January 1981

CURVATURE COLLINEATIONS FOR THE FIELD OF GRAVITATIONAL WAVES

K. P. SINGH AND GULAB SINGH

Department of Mathematics, Banaras Hindu University, Varanasi 221005

(*Received* 9 *August* 1979)

In this paper it has been shown that the space-times formed from a plane-fronted gravity wave and from a plane sandwich wave with constant polarisation admit proper curvature collineation in general. The curvature collineation vectors have been determined explicitly.

1. INTRODUCTION

Curvature collineation (CC) in general relativity signifies not only a type of geometrical symmetry of the space-time, but it also implies that the gravitational properties of the field are preserved along the CC vector. It has played an important role in the study of conservation laws. Katzin *et al.* (1969) have investigated various properties of geometrical and physical interest associated with CC vectors, and have established a number of theorems connecting various symmetry properties with curvature collineation.

A curvature collineation, admitted by a Riemannian space-time V_4 for an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta t$$

is defined by

$$\mathcal{L}_\xi R^h_{ijk} = 0$$

which can be expressed in terms of partial derivatives as

$$\mathcal{L}_\xi R^h_{ijk} = R^h_{ijk,m}\,\xi^m - R^m_{ijk}\,\xi^h_{,m} + R^h_{mjk}\,\xi^m_{,i} + R^h_{imk}\,\xi^m_{,j} + R^h_{ijm}\,\xi^m_{,k}$$
$$= 0 \qquad \text{...(1.1)}$$

where δt is a positive infinitesimal, $\mathcal{L}_\xi$ stands for the Lie derivative with respect to the field vector ξ^i and R^h_{ijk} is the Riemann curvature tensor. Throughout this paper the 'comma' and 'semicolon' followed by any suffix, or suffices, denote partial and covariant differentiation respectively with respect to the corresponding variables.

Besides the curvature collineation several other geometrical symmetries, such as motion, homothetic motion, conformal motion, affine collineation, projective collineation, etc. have been discussed by Katzin *et al.* (1969). We call a CC vetor proper if it does not admit higher symmetries such as those mentioned above. Recently Singh *et al.* (1978) examined the cylindrically symmetric field of total radiation obtained by Rao (1964), for CC and found that it admits proper curvature collineation.

In this paper the space-times representing a plane-fronted gravity wave (Rindler 1977) and colliding gravitational waves (Szekeres 1970) have been examined for CC and it has been found that the space-times admit curvature collineation in general.

2. A Plane Fronted Gravity Wave

It is expected that curvature disturbances in space-time would propagate with the speed of light giving rise to 'gravitational radiation'. This has been shown by many theoretical investigations, using mainly various approximation methods. However, there exist certain exact solutions of Einstein's vacuum field equations that clearly represent gravity waves. It is not the kind of wave that could be generated by any reasonable source; rather, it would have to be created in toto. It exhibits some interesting properties which might be expected to apply also in more general radiative situations. It is to be noted that a gravity wave must satisfy the vacuum field equations. Although it may carry gravitational energy, such energy is not an explicit source term in Einstein's theory and we shall assume that there are no other forms of energy (like mass, electromagnetic fields, etc.) in its path.

Any scalar wave profile $p = f(x)$ propagating with speed c in the x-direction will be given by the equation $p = f(x - ct)$ after time t. Here taking $c = 1$, we conventionally write $p = g(t - x)$, where $g(x) = f(-x)$. By the electromagnetic analogy, we expect gravitational waves to be transverse, i.e., to distort space-time only in directions perpendicular to the wave normal. Thus we are led to examine the space-time (Rindler 1977) representing an infinite 'plane-fronted sandwich wave'

$$ds^2 = dt^2 - dx^2 - p^2dy^2 - q^2dz^2 \qquad \text{...(2.1)}$$

where p, q are functions of $u = t - x$ alone and the coordinates x, y, z, t correspond to x^1, x^2, x^3, x^4 respectively.

For the metric (2.1), the non-vanishing components of the Christoffel symbol Γ^k_{ij} are

$$\left.\begin{aligned} \Gamma^1_{22} = \Gamma^4_{22} = -p^2\Gamma^2_{12} = p^2\Gamma^2_{24} = pp' \\ \Gamma^1_{33} = \Gamma^4_{33} = -q^2\Gamma^3_{13} = q^2\Gamma^3_{34} = qq' \end{aligned}\right\} \qquad \text{...(2.2)}$$

where 'dash' denotes ordinary differentiation with respect to u.

The non-zero components of the Riemann curvature tensor R^h_{ijk} are

$$\left.\begin{aligned} R^1_{212} &= R^1_{224} = R^4_{212} = -R^4_{242} = -pp'' \\ R^1_{313} &= R^1_{334} = R^4_{313} = -R^2_{343} = -qq'' \\ R^2_{121} &= R^2_{424} = R^2_{412} = -R^2_{124} = -p''/p \\ R^3_{131} &= R^3_{434} = R^3_{413} = -R^3_{134} = -q''/q. \end{aligned}\right\} \qquad \ldots(2.3)$$

From the algebraic symmetries on the indices we find that, in a V_4, eqn. (1.1) formally represents 96 equations. Considering these equations for the line element (2.1), we obtain the follwing set of equations :

The non-vanishing components of the Riemann curvature tensor give

$$\mathcal{L}_\xi R^1_{212} = 0 \Rightarrow (pp'')' (\xi^1 - \xi^4) + (pp'') (\xi^1_{,4} + \xi^4_{,1} - 2\xi^2_{,2}) = 0 \qquad \ldots(2.4)$$

$$\mathcal{L}_\xi R^1_{224} = 0 \Rightarrow (pp'')' (\xi^1 - \xi^4) + (pp'') (\xi^1_{,1} + 2\xi^1_{,4} - 2\xi^2_{,2} - \xi^4_{,4}) = 0 \qquad \ldots(2.5)$$

$$\mathcal{L}_\xi R^4_{212} = 0 \Rightarrow (pp'')' (\xi^1 - \xi^4) + (pp'') (\xi^4_{,4} + 2\xi^4_{,1} - 2\xi^2_{,2} - \xi^1_{,1}) = 0 \qquad \ldots(2.6)$$

$$\mathcal{L}_\xi R^1_{313} = 0 \Rightarrow (qq'')' (\xi^1 - \xi^4) + (qq'') (\xi^1_{,4} + \xi^4_{,1} - 2\xi^3_{,3}) = 0 \qquad \ldots(2.7)$$

$$\mathcal{L}_\xi R^1_{334} = 0 \Rightarrow (qq'')' (\xi^1 - \xi^4) + (qq'') (\xi^1_{,1} + 2\xi^1_{,4} - 2\xi^3_{,3} - \xi^4_{,4}) = 0 \qquad \ldots(2.8)$$

$$\mathcal{L}_\xi R^4_{313} = 0 \Rightarrow (qq'')' (\xi^1 - \xi^4) + (qq'') (\xi^4_{,4} + 2\xi^4_{,1} - 2\xi^3_{,3} - \xi^1_{,1}) = 0 \qquad \ldots(2.9)$$

$$\mathcal{L}_\xi R^2_{121} = 0 \Rightarrow (p''/p)' (\xi^1 - \xi^4) + (p''/p) (2\xi^4_{,1} - 2\xi^1_{,1}) = 0 \qquad \ldots(2.10)$$

$$\mathcal{L}_\xi R^2_{424} = 0 \Rightarrow (p''/p)' (\xi^1 - \xi^4) + (p''/p) (2\xi^1_{,4} - 2\xi^4_{,4}) = 0 \qquad \ldots(2.11)$$

$$\mathcal{L}_\xi R^2_{124} = 0 \Rightarrow (p''/p)' (\xi^1 - \xi^4) + (p''/p) (\xi^4_{,1} + \xi^1_{,4} - \xi^1_{,1} - \xi^4_{,4}) = 0 \qquad \ldots(2.12)$$

$$\mathcal{L}_\xi R^3_{131} = 0 \Rightarrow (q''/q)' (\xi^1 - \xi^4) + (q''/q) (2\xi^4_{,1} - 2\xi^1_{,1}) = 0 \qquad \ldots(2.13)$$

$$\mathcal{L}_\xi R^3_{434} = 0 \Rightarrow (q''/q)' (\xi^1 - \xi^4) + (q''/q) (2\xi^1_{,4} - 2\xi^4_{,4}) = 0 \qquad \ldots(2.14)$$

$$\mathcal{L}_\xi R^3_{134} = 0 \Rightarrow (q''/q)' (\xi^1 - \xi^4) + (q''/q) (\xi^4_{,1} + \xi^1_{,4} - \xi^1_{,1} - \xi^4_{,4}) = 0. \qquad \ldots(2.15)$$

The vanishing components of the Riemann curvature tensor provide

$$\mathcal{L}_\xi R^1_{214} = 0 \Rightarrow (pp'') (\xi^2_{,1} + \xi^2_{,4}) = 0 \qquad \ldots(2.16)$$

$$\mathcal{L}_\xi R^1_{223} = 0 \Rightarrow (pp'')(\xi^1_{,3} - \xi^4_{,3}) = 0 \qquad ...(2.17)$$

$$\mathcal{L}_\xi R^1_{314} = 0 \Rightarrow (qq'')(\xi^3_{,1} + \xi^3_{,4}) = 0 \qquad ...(2.18)$$

$$\mathcal{L}_\xi R^1_{323} = 0 \Rightarrow (qq'')(\xi^1_{,2} - \xi^4_{,2}) = 0 \qquad ...(2.19)$$

$$\mathcal{L}_\xi R^1_{112} = 0 \Rightarrow (p''/p)\,\xi^1_{,2} + (pp'')\,\xi^2_{,1} = 0 \qquad ...(2.20)$$

$$\mathcal{L}_\xi R^1_{412} = 0 \Rightarrow (p''/p)\,\xi^1_{,2} - (pp'')\,\xi^2_{,4} = 0 \qquad ...(2.21)$$

$$\mathcal{L}_\xi R^4_{112} = 0 \Rightarrow (p''/p)\,\xi^4_{,2} + (pp'')\,\xi^2_{,1} = 0 \qquad ...(2.22)$$

$$\mathcal{L}_\xi R^4_{412} = 0 \Rightarrow (p''/p)\,\xi^4_{,2} - (pp'')\,\xi^2_{,4} = 0 \qquad ...(2.23)$$

$$\mathcal{L}_\xi R^1_{113} = 0 \Rightarrow (q''/q)\,\xi^1_{,3} + (qq'')\,\xi^3_{,1} = 0 \qquad ...(2.24)$$

$$\mathcal{L}_\xi R^1_{413} = 0 \Rightarrow (q''/q)\,\xi^1_{,3} - (qq'')\,\xi^3_{,4} = 0 \qquad ...(2.25)$$

$$\mathcal{L}_\xi R^4_{113} = 0 \Rightarrow (q''/q)\,\xi^4_{,3} + (qq'')\,\xi^3_{,1} = 0 \qquad ...(2.26)$$

$$\mathcal{L}_\xi R^4_{413} = 0 \Rightarrow (q''/q)\,\xi^4_{,3} - (qq'')\,\xi^3_{,4} = 0 \qquad ...(2.27)$$

$$\mathcal{L}_\xi R^1_{213} = 0 \Rightarrow (qq'')\;\xi^3_{,2} + (pp'')\,\xi^2_{,3} = 0 \qquad ...(2.28)$$

$$\mathcal{L}_\xi R^2_{113} = 0 \Rightarrow \left(\frac{q''}{q} - \frac{p''}{p}\right)\xi^2_{,3} = 0. \qquad ...(2.29)$$

Here redundant and trivial equations have been omitted.

On imposing the condition $\xi^2_{,2} = \xi^3_{,3} = 0$ on the CC vector ξ^i, the set of eqns. (2.4) - (2.29) give

$$\xi^1_{,2} = \xi^4_{,2} = p^2\xi^2_{,4}\,,\ \xi^1_{,3} = \xi^4_{,3} = q^2\xi^3_{,4} \qquad ...(2.30)$$

$$\xi^2_{,2} = \xi^2_{,3} = \xi^2_{,1} + \xi^2_{,4} = 0 \qquad ...(2.31)$$

$$\xi^3_{,2} = \xi^3_{,3} = \xi^3_{,1} + \xi^3_{,4} = 0 \qquad ...(2.32)$$

$$\left.\begin{aligned} &\xi^1_{,2} = \xi^4_{,2} = -p^2\xi^2_{,1}\,,\ \xi^1_{,3} = \xi^4_{,3} = -q^2\xi^3_{,1} \\ &\xi^4_{,1} + \xi^4_{,4} = \xi^1_{,1} + \xi^1_{,4}\,. \end{aligned}\right\} \qquad ...(2.33)$$

In view of (2.31) and (2.32), the solutions for ξ^2 and ξ^3 can be put as follows:

$$\xi^2 = \alpha(t - x) \text{ and } \xi^3 = \beta(t - x) \quad \text{...(2.34)}$$

where α and β are arbitrary constants. Equations (2.30) and (2.33), in view of (2.34), give the solutions for ξ^1 and ξ^4 as

$$\xi^1 = \alpha p^2 y + \beta q^2 z = \xi^4. \quad \text{...(2.35)}$$

Now the set of eqns. (2.4) – (2.29) are satisfied for the values of ξ^i given by (2.34) and (2.35) under the condition $\xi^2_{,2} = \xi^3_{,3} = 0$.

Hence the space-time (2.1) corresponding to plane-fronted gravity wave admits the CC vector ξ^i given by (2.34) and (2.35).

Now we shall see that the CC vector ξ^i admitted by the metric (2.1) is proper. A CC which is not degenerate in the sense of being at the same time a stronger symmetry such as motion or affine collineation etc. is called proper. By the use of (2.34) and (2.35) and the definition $h_{ij} \equiv \mathcal{L}_\xi g_{ij} = \xi_{i;j} + \xi_{j;i}$, we see that

$$h_{11} = 4(\alpha pp'y + \beta qq'z) \neq 0;$$

which shows that the CC vector ξ^i does not define motion. For affine collineation ξ^i must satisfy $h_{ij;k} = 0$ in general. It is found that

$$h_{11;1} = 4 \frac{\partial}{\partial x} (\alpha pp'y + \beta qq'z) \neq 0.$$

Therefore ξ^i is not an affine collineation vector. It can easily be seen that $\xi^i_{;ijk} \neq 0$ in general, which shows that ξ^i does not define either a conformal motion (including homothetic motion) or projective and conformal collineations. Hence the CC vector ξ^i given by (2.34) and (2.35) is proper.

3. Colliding Gravitational Waves

We consider the space-time with co-ordinates $x^4 = u$, $x^1 = v$, $x^2 = x$, $x^3 = y$ where u and v are null coordinates ($u_\mu u_\nu g^{\mu\nu} = v_\mu v_\nu g^{\mu\nu} = 0$) and x and y are space-like co-ordinates. For $v < 0$ we will assume the metric to be that of a plane sandwhich wave with constant polarisation (Szekeres 1970)

$$ds^2 = 2e^{-M}\, du dv - e^{-U} (e^V dx^2 + e^{-V} dy^2) \quad \text{...(3.1)}$$

where M, U and V are functions of u alone subject to the differential equation

$$2U'' - U'^2 + 2U'M' = V'^2 \quad \text{...(3.2)}$$

where 'dash' denotes ordinary differentiation with respect to u.

For a sandwich wave we may assume space-time to be flat in region I ($u < 0$, $v < 0$) with $M = U = V = 0$ and for $u > 0$ (region II) the only independent surviving component of the Riemann curvature tensor (Szekeres 1970) is

$$R^3_{434} = -R^2_{424} = \tfrac{1}{2}\,[V'' + V'(M' - U')] \neq 0. \quad \text{...(3.3)}$$

For the metric (3.1), the non-zero components of the Christoffel symbol Γ^k_{ij} are

$$\left.\begin{aligned} \Gamma^2_{24} &= e^{U-V-M}\Gamma^1_{22} = \tfrac{1}{2}(V' - U') \\ \Gamma^3_{34} &= e^{U+V-M}\Gamma^1_{33} = -\tfrac{1}{2}(V' + U') \\ \Gamma^4_{44} &= -M'. \end{aligned}\right\} \quad \text{...(3.4)}$$

Now eqn. (1.1), for the metric (3.1), gives the following set of equations for the non-vanishing and vanishing components of the Riemann curvature tensor:

$$\mathcal{L}_\xi R^2_{424} = 0 \Rightarrow P_{,4}\xi^4 + 2P\xi^4_{,4} = 0 \quad \text{...(3.5)}$$

$$\mathcal{L}_\xi R^3_{434} = 0 \Rightarrow P_{,4}\xi^4 + 2P\xi^4_{,4} = 0 \quad \text{...(3.6)}$$

where $P = -\tfrac{1}{2}\,[V'' + V'((M' - U')]$.

$$\mathcal{L}_\xi R^1_{442} = 0 \Rightarrow \xi^1_{,2} = 0 \quad \text{...(3.7)}$$

$$\mathcal{L}_\xi R^4_{424} = 0 \Rightarrow \xi^4_{,2} = 0 \quad \text{...(3.8)}$$

$$\mathcal{L}_\xi R^3_{414} = 0 \Rightarrow \xi^3_{,1} = 0 \quad \text{...(3.9)}$$

$$\mathcal{L}_\xi R^1_{443} = 0 \Rightarrow \xi^1_{,3} = 0 \quad \text{...(3.10)}$$

$$\mathcal{L}_\xi R^2_{441} = 0 \Rightarrow \xi^2_{,1} = 0 \quad \text{...(3.11)}$$

$$\mathcal{L}_\xi R^2_{423} = 0 \Rightarrow \xi^4_{,3} = 0 \quad \text{...(3.12)}$$

$$\mathcal{L}_\xi R^3_{442} = 0 \Rightarrow \xi^3_{,2} = 0 \quad \text{...(3.13)}$$

$$\mathcal{L}_\xi R^2_{124} = 0 \Rightarrow \xi^4_{,1} = 0 \quad \text{...(3.14)}$$

$$\mathcal{L}_\xi R^3_{444} = 0 \Rightarrow \xi^3_{,4} = 0 \quad \text{...(3.15)}$$

$$\mathcal{L}_\xi R^2_{444} = 0 \Rightarrow \xi^2_{,4} = 0 \quad \text{...(3.16)}$$

$$\mathcal{L}_\xi R^2_{443} = 0 \Rightarrow \xi^2_{,3} = 0. \quad \text{...(3.17)}$$

Redundant and trivial equations have been omitted. By inspection we find, from eqns. (3.5) – (3.17), that

$$\xi^1_{,2} = \xi^1_{,3} = 0 \quad \text{...(3.18)}$$

$$\xi^2_{,1} = \xi^2_{,3} = \xi^2_{,4} = 0 \quad ...(3.19)$$

$$\xi^3_{,1} = \xi^3_{,2} = \xi^3_{,4} = 0 \quad ...(3.20)$$

$$\xi^4_{,1} = \xi^4_{,2} = \xi^4_{,3} = 0 \quad ...(3.21)$$

$$P_{,4}\xi^4 + 2P\xi^4_{,4} = 0.$$

Equations (3.21) will give the solution for ξ^4 as

$$\xi^4 = \frac{\alpha}{P^{1/2}} \quad ...(3.22)$$

where α is an integration constant. In view of (3.18), (3.19) and (3.20), we can put the solutions for ξ^1, ξ^2 and ξ^3 as

$$\left.\begin{aligned} \xi^1 &= F(u, v) \\ \xi^2 &= G(x) \\ \xi^3 &= H(y) \end{aligned}\right\} \quad ...(3.23)$$

where F, G and H are arbitrary functions of their arguments.

Thus the space-time (3.1) corresponding to plane sandwich wave with constant polarisation admits the CC vector ξ^i given by (3.23) and (3.22).

It has been verified that the collineation vector ξ^i given by (3.23) and (3.22) satisfies the following equations

$$h_{ij} \equiv \mathcal{L}_\xi g_{ij} \neq 0 \quad \text{in general}$$

and

$$\mathcal{L}_\xi \Gamma^k_{ij} \neq 0 \quad \text{in general}$$

which proves that the collineation is a proper CC.

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