Indian J. pure appl. Math., 12(2) : 151–162, February 1981

AN ALTERNATIVE APPROACH FOR SOLVING A CERTAIN CLASS OF TIME OPTIMAL CONTROL PROBLEMS

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(*Received* 7 *September* 1977; *after revision* 15 *September* 1980)

This paper deals with the problem of optimum control of linear systems for minimum time when the control function is subject to a certain class of constraints. The nature of the constraints is such that it is possible to identify a Banach space to which the control function belongs and the constraints on the control can be expressed as a constraint on the appropriate norm. One then considers the auxilliary problem of minimising this norm for a terminal time given in advance, which is solved by functional analytic techniques. It has been demonstrated how the solution of the original problem is obtained from that of the auxilliary problem. The advantage of this approach is that it is applicable in situations where Pontryagin maximum principle cannot be applied. One example has been included to illustrate the application of the theory.

We shall consider the nth order system satisfying the following vector matrix differential equation

$$\frac{dX}{dt} = A(t)\,X(t) + B(t)\,U(t) \qquad \ldots(1)$$

where X is an n-vector representing the state of control system at time t, $U(t)$ is an r-vector $(r \leqslant n)$ representing the control input of the system, $A(t)$ an $(n \times n)$ matrix and $B(t)$ an $(n \times r)$ matrix.

The time optimal control problem regarding such systems is a classical one in the theory of control. The problem reads as follows:

"Find the control variables, constrained in some manner, which bring the state of the controlled plant from some initial state to a desired final one in minimum time".

The investigation of this problem was pioneered by Bushaw (1958) and independently by Feldbaum (1953). Bushaw considered second order autonomous linear systems with an amplitude constrained control variable. He proved that the relay or the bang-bang control is optimal and constructed the switching curves in the phase plane. Feldbaum proved the same for more general cases and showed that at most $(n - 1)$ switchings of sign of the optimal bang-bang control are

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necessary, in the case of nth order systems with negative, real and distinct eigen values while in the case of complex roots, the number of switchings depends entirely on the initial position of the system.

This was also proved independently by Bellman *et al.* (1956) by a more general and powerful method.

Pontryagin *et al.* (1962), near about the same time, also developed the "Maximum Principle" to tackle such problems. Chaudhuri (1963) and Chaudhuri and Choudhury (1964) derived the optimal bang-bang control and pointed out that the switching functions can be obtained from the solution of the adjoint-system. Krasovskii (1957, 1959) applied methods of functional analysis in obtaining his results. Kulikowski (1959a, b), Kranc and Sarachik (1963), and Kreindler (1963) worked in this region with various types of constraints on the control variables. Hermes and Lasalle (1969) also considered time optimal control problem from functional analysis point of view for amplitude constraints. Krasovskii pointed out that the solution to the time optimal control problem implies the solution of the fixed terminal time in minimum-norm problem. Porter and Williams (1966a, b) and Porter (1966) demonstrated as to how the function space approach could be utilised to determine the optimal control for a wide class of minimum-norm problems which would otherwise be difficult to obtain. Minamide and Nakamura (1971) considered the minimum cost problem of Porter in a wider sense. Burns (1975a, b) also considered the minimum effort problem and minimum cost problem in the Banach space setting. The main purpose of this paper is to demonstrate how the switching functions of a wide class of problems can be very conveniently determined from the structure of the reachable set. This approach essentially derives its validity from the fact that it is possible to identify the minimum time control with that for the corresponding minimum-norm control. We first establish this identification for general spaces and then proceed to illustrate the application of the approach to a simple control problem. The time optimal control problem can be defined as follows:

Let B_t be a Banach space depending upon the continuous parameter t, where t denotes the time. Let D be another Banach space. Let T_t be a transformation depending upon the parameter t, mapping B_t onto D. Let $U_t \subset B_t$ be the unit ball in B_t and $\xi \in D$. The problem is to determine $u \in U_t$ such that $T_t u = \xi$ and t is minimum. Here we shall consider only the case when T_t is linear and bounded and onto.

In the problems which usually arise in practice, B_t is an increasing function of t in the sense that $B_{t_1} \subset B_{t_2}$ whenever $t_1 \leqslant t_2$. Also T_{t_1} can be regarded as the restriction of T_{t_2} defined on B_{t_2}, on B_{t_1}. It is easy to show that under the above conditions $U_{t_1} \subset U_{t_2}$.

The necessary and sufficient conditions for controllability has already been discussed for the minimum time problem in the special case when B_t coincides with $L_\infty(0, t)$ and D is n-dimensional Euclidean space. The necessary and sufficient condition for controllability for more general Banach spaces will be discussed elsewhere.

Reachable Region (Set)

The set of all points $\xi \in D$ such that $T_t u = \xi$ for all $u \in U_t$ will be called the Reachable Region (Set) with respect to the linear transformation T_t and will be denoted by $C(t)$. The following theorem can be verified easily (Porter 1966).

Theorem 1 — The reachable region is bounded and a convex body, symmetrical with respect to the origin of D.

Corollary — The reachable region $C(t)$ is closed when B_t is either a reflexive space or it can be considered as a conjugate of some other Banach space.

PROOF : It is known that a space is reflexive if and only if its unit ball is weakly compact (Hille and Philips 1957). So, if it is assumed that B_t is reflexive, then the unit ball U_t is weakly compact. Again, since T_t is linear and bounded, it is continuous. Also, the continuous image of weakly compact set is weakly compact. Consequently $C(t)$ is weakly compact and hence it is weakly closed and therefore it is strongly closed. Then $C(t)$ is closed.

However, if B_t is not a reflexive space but it can be considered as a conjugate of some other Banach space, then by Alaglu's theorem it follows that its unit ball U_t is weakly compact in some topology. Therefore by the previous analogy we can conclude that $C(t)$ is closed in this case also.

To solve the minimum time control problem we shall first consider the following auxiliary problem.

Auxilliary problem — Let $\xi \in \delta C(t)$ where $\delta C(t)$ denotes the boundary of $C(t)$ for some given time t. Then determine $u \in U_t$ such that $T_t u = \xi$ and $\| u \|$ is minimum. We shall call this the minimum norm problem. The corresponding control will be called the optimal control.

In the following theorems we shall find the form of the optimal control and also the shape of the reachable set w.r.t. the minimum time t.

Theorem 2 — Let $\xi \in \delta C(t)$ where t is the given terminal time, and $\phi \in D^*$ define the supporting hyperplane at ξ. Let u_ϕ be the optimal control to reach at ξ in the above sense. Then u_ϕ maximises $\langle u, T_t^* \phi$ where T_t^* and D^* denote the adjoint transformation and adjoint space to T_t and D respectively.

To prove the above theorem we require the following two lemmas.

Lemma 1 — An admissible control which will be optimal in the above sense must satisfy $\| u \| = 1$.

PROOF : We have already shown that $C(t)$ is a closed convex body. Thus if $\xi \in \delta C(t)$, then there exists a $\phi \in D^*$, where D^* is the conjugate space to D, such that $\langle \xi, \phi \rangle \geqslant \langle \eta, \phi \rangle$ for all $\eta \in C(t)$. Let $u \in U_t$ be such that $T_t u = \eta$. Since $C(t)$ is circled, it follows, that $\langle \xi, \phi \rangle \geqslant | \langle T_t u, \phi \rangle | = | \langle u, T_t^* \phi \rangle |$ for all $u \in U_t$, T_t^* being the transformation adjoint to T_t.

$$\therefore \quad \langle \xi, \phi \rangle \geqslant \sup_{\|u\| \leqslant 1} | \langle u, T_t^* \phi \rangle | = \| T_t^* \phi \|.$$

Now let $u \in U_t$ such that $T_t u = \xi$.

Then $\| u \| \, \| T_t^* \phi \| \geqslant \langle u, T_t^* \phi \rangle = \langle T_t u, \phi \rangle = \langle \xi, \phi \rangle \geqslant \| T_t^* \phi \|$. $\therefore \| u \| \geqslant 1$

Thus $\| u \| = 1$. This proves the lemma.

Lemma 2 — Let $\xi \in \delta C(t)$ and $\phi \in D^*$ determine the supporting hyperplane to $C(t)$ at ξ. Then $\langle \xi, \phi \rangle = \| T_t^* \phi \|$.

PROOF : Since $\xi \in \delta C(t)$, there is a $u_\phi \in U_t$ such that $\xi = T_t u_\phi$. Hence by Lemma 1

$$\langle \xi, \phi \rangle \geqslant \| T_t^* \phi \|. \qquad \ldots(1)$$

Also $$\langle \xi, \phi \rangle = \langle T_t u_\phi, \phi \rangle = \langle u_\phi, T_t^* \phi \rangle \leqslant \| u_\phi \| \, \| T_t^* \phi \| \leqslant \| T_t^* \phi \| \qquad \ldots(2)$$

from (1) and (2) $\langle \xi, \phi \rangle = \| T_t^* \phi \|$.

Again, as $0 \in \operatorname{Int} C(t)$ ($C(t)$ is a convex body), it follows that $\langle \xi, \phi \rangle > 0$.

Proof of Theorem 2 — $\phi \in D^*$ can be chosen so that $\langle \eta, \phi \rangle \leqslant \langle \xi, \phi \rangle$ for all $\eta \in C(t)$. Let $\eta = T_t u$ then $\langle u, T_t^* \phi \rangle \leqslant \langle u_\phi, T_t^* \phi \rangle$ which proves the Theorem.

Also, we have $\langle u_\phi, T_t^* \phi \rangle \leqslant \| u_\phi \| \, \| T_t^* \phi \| \leqslant \| T_t^* \phi \|$.

But by Hahn-Banach theorem — there exists, a u_ϕ such that $\langle u_\phi, T_t^* \phi \rangle = \| T_t^* \phi \|$ and $\| u_\phi \| = 1$. Hence the maximum is actually attained at u_ϕ. u_ϕ is called the extremal of $T_t^* \phi$ and is denoted by $\overline{T_t^* \phi}$ i.e. $u_\phi = \overline{T_t^* \phi}$. From this follows, if $\xi \in \delta C(t)$ the time optimal control to drive the system from the origin to ξ will be given by

$u_\phi = \overline{T_t^* \phi}$, where $\phi \in D^*$ defines the supporting hyperplane to $\delta C(t)$ at ξ. Thus solution of the minimum norm control problem leads to the solution of the time optimal control problem. This procedure of solving the time optimal control problem will evidently be valid so long as $C(t)$ is an increasing function of t in the sense $C(t_1) \subset C(t_2)$ whenever $t_1 < t_2$, and $\delta C(t_1) \cap \delta C(t_2) = \Phi$ and assume global controllability i.e. given any point $\xi \in D$, there exists a t such that $\xi \in \delta C(t)$. In what follows we have finally proved a necessary and sufficient condition for $C(t)$ to be a strictly increasing function of t, (see Theorem 5). The application of this approach has also been demonstrated with the help of a simple example. We shall prove the following theorems.

Theorem 3 — Let K be a weakly compact, convex set in a Banach space D and let ϕ be any element $\in D^*$, the conjugate space to D. Then there exists a point $\eta_0 \in K$, such that ϕ defines a supporting hyperplane to K at $\eta_0 \in \delta K$.

PROOF : If ϕ is a supporting hyperplane to K at η_0 then the theorem is proved. So let us suppose that ϕ is not a supporting hyperplane at any $\xi \in K$. Now, because $\phi \in D^*$ and K is bounded, therefore $\langle \eta, \phi \rangle \leqslant C$ for all $\eta \in K$ where $C > 0$ is some constant. Thus $\sup_{\eta \in K} \langle \eta, \phi \rangle$ will exist. Put $\sup_{\eta \in K} \langle \eta, \phi \rangle = M$. Then there will exist a sequence $\{\eta_n; \eta_n \in K\}$ such that $\langle \eta_n, \phi \rangle > M - (1/n)$ for $n \geqslant N$. Again, since K is weakly compact, there will exist a subsequence $\{\eta_{n_k}\}$ such that $\{\eta_{n_k}\}$ converges weakly to some $\eta_0 \in K$.

Therefore $\langle \eta_{n_k}, \phi \rangle > M - \dfrac{1}{n_k}$. Therefore $\lim \langle \eta_{n_k}, \phi \rangle = \langle \eta_0, \phi \rangle \geqslant M$.

On the other hand, $\langle \eta_{n_k}, \phi \rangle \leqslant M$ (because M is the least upper bound). Therefore $\lim \langle \eta_{n_k}, \phi \rangle = \langle \eta_0, \phi \rangle \leqslant M$. Hence $\langle \eta_0, \phi \rangle = M$.

Thus $\langle \eta, \phi \rangle \leqslant \langle \eta_0, \phi \rangle$ for all $\eta \in K$.

Also the above relation shows that the functional ϕ assumes its maximum value on K at the vector η_0, which together with the fact that any linear functional maps open sets into open sets, shows that η_0 cannot belong to interior of K. Consequently $\eta_0 \in \delta K$.

Theorem 4 — Let $\xi \in C(t_1) \cap \delta C(t_1)$ where $C(t_1)$ is the reachable region. Then

$$\operatorname*{Max}_{\psi} \frac{\langle \xi, \psi \rangle}{\| T_{t_2}^* \psi \|} \text{ is } \leqslant 1 \text{ or } \geqslant 1 \text{ according as } t_2 \geqslant t_1 \text{ or } t_2 \leqslant t_1$$

(here B_t is to be considered as in Corollary of Theorem 1).

To prove this we require the following lemma.

Lemma — Let $\xi \in \delta C(t_1)$, $t_2 > t_1$. Then the ray $k\xi$, $k > 0$ intersects $\delta C(t_2)$ at some point $\eta = l\xi$, $l \geqslant 1$.

PROOF : Since $C(t_2)$ is bounded (by Theorem 1), there will exist a $k > 0$, say $k = k_0$, such that $k_0\xi \notin C(t_2)$. Consider the portion of the ray $R = [k\xi, 0 \leqslant k \leqslant k_0]$. We now consider a set S defined by $S = \{k : k\xi \in C(t_2)\}$. Let $l = \sup_{k \in S} k$ which will evidently exist (because $k \leqslant k_0$). Evidently $l \geqslant 1$. Now there exists a sequence $\{k_n\}$ such that $\lim k_n = l$ and $k_n\xi = x_n \in R \cap C(t_2)$. Again, since R is compact, there is a subsequence $\{x_{n_k}\}$ such that limit $x_{n_k} = x_0 \in R$. Also, as $x_{n_k} \in C(t_2)$ and $C(t_2)$ is closed, therefore $x_0 = l\xi \in C(t_2)$. Now, $x_0 \notin \text{Int } C(t_2)$, because, if $x_0 \in \text{Int } C(t_2)$ then there will be an open sphere S_ϵ of radius ϵ which will be contained entirely within $C(t_2)$. Consider the point $x = x_0 + \frac{\epsilon}{2} \frac{\xi}{\| \xi \|}$. Then, $x \in S_\epsilon$ and $x \in R$. But then $x = \left\{ l + \frac{\epsilon}{2 \| \xi \|} \right\} \xi$, which contradicts the fact that $l = \sup_{k \in S} k$. This completes the proof of the lemma.

Proof of Theorem 4 — We shall prove the theorem for $t_2 \geqslant t_1$. Then we are required to show that $\text{Max}_{\psi} \frac{\langle \xi, \psi \rangle}{\| T^*_{t_2} \psi \|} \leqslant 1$ for a given $\xi \in C(t_1) \cap \delta C(t_1)$. Now, because $t_2 \geqslant t_1$, we have $B_{t_1} \subset B_{t_2}$ and $U_{t_1} \subset U_{t_2}$ (by assumption). The transformation T_{t_2} is such that T_{t_1} is the restriction of T_{t_2} on U_{t_1}.

Hence $\qquad C(t_1) = T_{t_1} U_{t_1} = T_{t_2} U_{t_1} \subset T_{t_2} U_{t_2} = C(t_2)$ therefore $\xi \in C(t_2)$.

Let $\psi \in D^*$, where D^* is the conjugate space to the Banach space D. Consequently by Theorem 3, there exists a point $\xi' \in \delta C(t_2)$, such that ψ defines a supporting hyperplane to $C(t_2)$ at ξ'. Again, since $\xi' \in \delta C(t_2)$ and ψ defines at supporting hyperplane to $C(t_2)$ at ξ', hence we can write $\langle \xi', \psi \rangle = \| T^*_{t_2} \psi \|$ (by Lemma 2 of Theorem 2). But $\langle \xi, \psi \rangle \leqslant \langle \xi', \psi \rangle$ as ψ defines a supporting hyperplane at $\xi' \in \delta C(t_2)$.

Therefore $\quad \langle \xi, \psi \rangle \leqslant \langle \xi', \psi \rangle = \| T^*_{t_2} \psi \|$ such that $\frac{\langle \xi, \psi \rangle}{\| T^*_{t_2} \psi \|} \leqslant 1$.

Hence
$$\sup_{\psi} \frac{\langle \xi, \psi \rangle}{\| T^*_{t_2} \psi \|} \leqslant 1.$$

Now $\xi \in \delta C(t_1)$ and let $\eta = l\xi \in \delta C(t_2)$ $(t_2 \geqslant t_1)$ (by the above lemma). Let ψ define the supporting hyperplane to $\delta C(t_2)$ at η.

Hence by Lemma 2 of Theorem 2, we get $\frac{\langle \eta, \psi \rangle}{\| T^*_{t_2} \psi \|} = 1$. Therefore $\frac{\langle \xi, \psi \rangle}{\| T^*_{t_2} \psi \|} = \frac{1}{l} \leqslant 1$.

Consequently sup is attained at a point $\phi = \psi \in D^*$, where ψ defines the supporting hyperplane at $\eta = l\xi \in \delta C(t_2)$. Thus we have proved the theorem for $t_2 \geqslant t_1$. Similarly, we can show that if $t_2 \leqslant t_1$ then $\underset{\phi}{\text{Max}} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_2}\phi \|} \geqslant 1$. This completes the proof of the theorem.

Theorem 5 — Let $t_1 < t_2$ and $T_{t_1} : B_{t_1} \to D$, $T_{t_2} : B_{t_2} \to D$ be bounded linear onto transformations. Then $C(t_1) \subseteq C(t_2)$ and $\delta C(t_1) \cap \delta C(t_2) = \Phi$ iff $\| T^*_{t_2}\phi \| > \| T^*_{t_1}\phi \|$, $\phi \in D^*$ and Φ denotes the null set.

(Here B_{t_1} and B_{t_2} are to be considered as in corollary of Theorem 1).

PROOF : We have already assumed that if $t_1 < t_2$, then $B_{t_1} \subseteq B_{t_2}$. Let U_{t_1} and U_{t_2} denote the unit balls in B_{t_1} and B_{t_2} respectively. Let $C(t_1)$ and $C(t_2)$ be reachable regions in D with respect to the transformations T_{t_1} and T_{t_2} corresponding to the times t_1 and t_2 respectively. Let $\xi \in C(t_1)$. Then there exists $u_1 \in U_{t_1}$ such that $T_{t_1}(u_1) = \xi$. As $u_1 \in B_{t_1} \subset B_{t_2}$, hence $T_{t_2}(u_1) \in C(t_2)$.

But $T_{t_1}(u_1) = T_{t_2}(u_1)$ as T_{t_1} is the restriction of T_{t_2} on B_{t_1}.

Hence $\quad \xi \in C(t_2)$. $\therefore$ $C(t_1) \subseteq C(t_2)$.

To prove the next part, let us assume that $\| T^*_{t_2}\phi \| > \| T^*_{t_1}\phi \|$. We shall show that $\delta C(t_1) \cap \delta C(t_2) = \Phi$. Let $\phi \in D^*$ be a functional over D. Then corresponding to ϕ we can find by Theorem 3, a point $\xi \in \delta C(t_1)$ and a point $\eta \in \delta C(t_2)$, such that ϕ is the supporting hyperplane at ξ to $C(t_1)$ and at η to $C(t_2)$ (Because according to the conditions of the theorems $C(t_1)$ and $C(t_2)$ are weakly compact and convex sets).

Also $\langle \eta, \phi \rangle = \| T^*_{t_2}\phi \|$ and $\langle \xi, \phi \rangle = \| T^*_{t_1}\phi \|$ (by Lemma 2 of Theorem 2).

Again by the Lemma of Theorem 4, corresponding to $\eta \in \delta C(t_2)$, we can find a point $\xi' \in \delta C(t_1)$ such that $\xi' = l\eta$ where $l \leqslant 1$.

Now, we have, $\| T^*_{t_2}\phi \| > \| T^*_{t_1}\phi \|$ (by hypothesis).

Hence $\langle \eta, \phi \rangle > \langle \xi, \phi \rangle$. Also, $\langle \xi, \phi \rangle \geqslant \langle \xi', \phi \rangle$, since ϕ is the supporting hyperplane to ξ and ξ' is any point in $\delta C(t_1)$. Consequently $\langle \eta, \phi \rangle > \langle \xi', \phi$ i.e. $\langle \eta, \phi \rangle > \langle l\eta, \phi \rangle = l \langle \eta, \phi \rangle$ (Because $\xi' = l\eta$, $l \leqslant 1$). Hence $\delta C(t_1) \cap \delta C(t_2) = \Phi$.

Conversely let $\delta C(t_1) \cap \delta C(t_2) = \Phi$. We are to show $\| T^*_{t_2}\phi \| > \| T^*_{t_1}\phi \|$ for $t_1 < t_2$ and for all $\phi \in D^*$. Let $\phi \in D^*$, be any functional over D. Hence corresponding to ϕ we can find by Theorem 3, a point $\xi \in \delta C(t_1)$ and a point $\eta \in \delta C(t_2)$ such that ϕ is the supporting hyperplane at $\xi \in C(t_1)$ and at $\eta \in C(t_2)$. Then by Lemma 2 of Theorem 2, $\langle \xi, \phi \rangle = \| T^*_{t_1}\phi \|$, $\langle \eta, \phi \rangle = \| T^*_{t_2}\phi \|$. Now because $C(t_1) \subseteq C(t_2)$ and by hypothesis $\delta C(t_1) \cap \delta C(t_2) = \Phi$, hence $\xi \in \text{Int}\, C(t_2)$. Thus $\langle \xi, \phi \rangle < \langle \eta, \phi \rangle$ where ϕ is the supporting hyperplane at η to $C(t_2)$.

Therefore $\quad \| T^*_{t_1}\phi \| < \| T^*_{t_2}\phi \|$.

The following corollary can be proved.

Corollary — If $\delta C(t_1) \cap \delta C(t_2) = \Phi$ then $\| T_{t_1} \| < \| T_{t_2} \|$.

PROOF : We have

$$\| T^*_{t_1}\phi \| < \| T^*_{t_2}\phi \| \leqslant \| T^*_{t_2} \| \, \| \phi \|$$

Therefore $\quad \| T^*_{t_1} \| < \| T^*_{t_2} \|$. Hence, $\| T_{t_1} \| < \| T_{t_2} \|$.

The above theorem justifies the identification of the minimum time control with that for minimum norm problem involving linear bounded onto transformations.

The solution of the system (1) is given by (assuming $A(t)$ and $B(t)$ are independent of t),

$$X(t) = e^{-At}X(0) + e^{-At} \int_0^t e^{As}Bu(S)\, dS.$$

or
$$e^{At}X(t) - X(0) = \int_0^t e^{As}Bu(S)\, dS$$

where $X(0)$ is the initial condition and t is the time for which the system is allowed to run.

Putting $e^{At}X(t) - X(0) = \xi$, we have

$$\xi = \int_0^t e^{As}Bu(S)\, dS = T_t u_t,\ u \in U_t \qquad \ldots(2)$$

where U_t denotes the unit ball in some Banach space and T_t is a linear onto transformation. The set of all points satisfying (2) for all $u \in U_t$ is the reachable Region corresponding to the time t. We shall consider the application of the above concept to the specific dynamical system which is governed by the following system of equations (Porter 1966):

$$\dot{X}_1 = X_2 + u_1,\quad \dot{X}_2 = u_2 \text{ where } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore $e^{-At}X(t) - X(0) = \int_0^t e^{-As}Bu(S)\,dS = \int_0^t \begin{bmatrix} 1 & -S \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dS = T_t u.$

Therefore $\xi_1 = \int_0^t (u_1 - Su_2)\,dS$ and $\xi_2 = \int_0^t u_2\,dS$

T_t being the transformation sending U_t into R^2.

It can be easily verified that T_t is onto.

Consider the circumstance in which u_1 and u_2 represent flow rates from independent fuel supplies for which saturation sets in at the same value for each supply. If the fuel supply itself is unlimited, it is natural to choose as an optimality criterion the function

$$J(u) = \sup_{t \in [0, t]} \max \{ | u_1(t) | , | u_2(t) | \} \quad \text{(Porter 1966, p. 317).}$$

Without loss of engineering significance, we require that u_1, u_2 be measurable. $J(u)$ then coincides with the norm of the tuplet (u_1, u_2) as an element of the space $L_\infty(l_\infty(2), \tau)$ which will be denoted by $B_{\infty,\infty}$. We shall first determine the boundaries (Isochrones) of the reachable set with respect to the mapping $T_t : B_{\infty,\infty} \rightarrow R^2$.

Now, $B_{\infty,\infty}$ is the conjugate of the Banach space $B_{1,1}$ i.e. $B^*_{1,1} = B_{\infty,\infty}$.

Let us consider a linear transformation $S : R^2 \rightarrow B_{1,1}$ (Porter 1966, p. 318)

Now corresponding to S there exists a linear transformation $S^* : B_{\infty,\infty} \rightarrow R^2$.

Hence $\quad T_t = S^*$.

Again by Alaglu's theorem it follows that the unit ball in the dual of any Banach space is weakly compact. Thus the unit ball in $B_{\infty,\infty}$ is weakly compact. Hence it is weakly closed and therefore it is strongly closed. Also, $S^{**}: R^2 \rightarrow B^*_{\infty,\infty}$. Now, $B_{1,1}$ is isomorphic to a closed subspace of $B^{**}_{1,1}$, and S^{**} would coincide with $S: R^2 \rightarrow B_{1,1}$ and $T^*_t = S^{**} = S$. If $\phi = (\phi_1, \phi_2) \in R^2$ then $T^*_t\phi \in B_{1,1}$ and $\overline{T^*_t\phi} \in B^*_{1,1} = B_{\infty,\infty}$.

It can also be proved that the image of the unit ball of $B_{\infty,\infty}$ is closed (Porter 1966). It can be verified that $\| T^*_{t_1} \phi \| < \| T^*_{t_2} \phi \|$, $t_1 < t_2$ for all $\phi \in D^*$, in this problem. In view of the above, the optimal control is given by the following (Porter 1966):

$$u_1(t) = \begin{cases} \text{Sign}\,[\phi_1], & \phi_1 \neq 0 \\ | u_1(t) | \leqslant 1, & \phi_1 = 0 \end{cases} \qquad t \in [0, t]$$

$$u_2(t) = \text{Sign}\,[\phi_2 - \phi_1 t],\ t \in [0, t].$$

Since the set $C(t)$ is closed, $\delta C(t)$ may be traced out by computing the points

$$\{T_t(\overline{T_t^* \phi}) \ : \phi \in R^2\}.$$

The structure of the reachable set is depicted in Fig. 1. The control sequence which has been used in the foregoing, is needed to go from the origin of the state plane to some point on the boundary of the reachable set in minimum time.

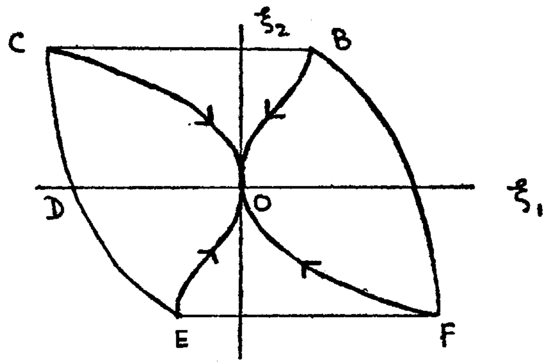

FIG. 1.

Now our object is to drive any initial state to (0, 0), i.e. to the origin of the state plane in minimum time. Let (ξ_1, ξ_2) be any initial state. Consider the reachable set such that (ξ_1, ξ_2) lies on the boundary of the set corresponding to the time t. If we now give a translation to reachable set through the point $\xi = (\xi_1, \xi_2)$ we shall get the same reachable set with $\xi = (\xi_1, \xi_2)$ as centre corresponding to the time t, which will also be convex (any translate of a convex set is convex). Thus the time taken to reach any point on the boundary from (ξ_1, ξ_2) in the translated convex body, is the same as that would be required to go from (0, 0) to corresponding point on the boundary of the former convex body. The optimal control will also be identical in both the cases. Since the reachable set is symmetrical about origin, the optimal control required to go from (0, 0) to $(-\xi_1, -\xi_2)$ will be $-u$, where u is the optimal control required to drive the system from the origin (0, 0) to $\xi = (\xi_1, \xi_2)$. The point

$$-\xi = (-\xi_1, -\xi_2)$$

coincides with the origin of the phase plane after translation which now lies on the boundary of the reachable set with respect to $\xi = (\xi_1, \xi_2)$ as centre. The optimal control to go from $\xi = (\xi_1, \xi_2)$ to the origin will be determined by the direction cosines of the outward drawn normal to the supporting hyperplane to the translated convex set (the reachable set with respect to (ξ_1, ξ_2) as centre) at the origin of the phase space.

The direction cosines of this normal will be the same as those at $-\xi$ with respect to the original reachable set about the origin of the phase space, since it is invariant under linear transformation. Hence the optimal control to drive the system from $\xi = (\xi_1, \xi_2)$ to origin will be the same as that would be required to drive the system from $(0, 0)$ to $-\xi = (-\xi_1, -\xi_2)$. It can be shown that the reachable region is divided into four regions namely *OBF*, *OBC* and their reflections (*i.e.* *OEC*, *EOF*). The important features of the above regions are as follows:

(1) *OB* : is a trajectory but not a switching curve. It divides the regions of switching and no switching.

(2) *OF*: is a switching curve and also a trajectory. Similarly *OE* and *OC* which are reflections of *OB* and *OF* respectively, can be identified as above. The optimal control law will be as follows:—

$$u^* = (-1, -1) \text{ for all } (\xi_1, \xi_2) \in \text{Region } \left[OBF : \xi_2 - \frac{\xi_2^2}{2} < \xi_1 \leqslant -\xi_2 + \frac{\xi_2^2}{2}, \xi_1 > 0 \right]$$

$$u^* = \left(-\frac{\xi_1 + \frac{1}{2}\xi_2^2}{\xi_2}, -1 \right) \text{ for all } (\xi_1, \xi_2) \in \text{Region}$$

$$\left[OBC : -\xi_2 - \frac{\xi_2^2}{2} < \xi_1 < \xi_2 - \frac{\xi_2^2}{2}, \xi_2 > 0 \right].$$

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