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SOLUTION OF THE CONJUGATE BOUNDARY VALUE PROBLEM OF MASS TRANSFER IN CIRCULAR DUCT DIALYSERS BY USING GALERKIN'S METHOD

J. N. KAPUR, F.N.A.

Department of Mathematics, Indian Institute of Technology, Kanpur 208016

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An alternative technique, based on Galerkin's method, for solving the conjugate boundary value problem arising from diffusion, convection and mass transfer in dialysers is given and compared with the earlier one of Cooney *et al.* (1974) which was based on expansion in terms of eigenfunctions developed as a modification of method used still earlier by Nunge and Gill (1966) for their analysis of laminar flow in heat exchangers.

1. INTRODUCTION

Most of the literature before 1974 on mass transfer in dialysers assumed that (a) the concentration of the solute in the dialyzate is zero or constant everywhere and that (b) the mass transfer resistance in the dialyzate is constant and may be added linearly to the resistance of the membrane to give a composite resistance. Cooney *et al.* (1974) solved the conjugate boundary value problem of solving the two convective diffusion partial differential equations for the blood and dialyzate regions, coupled by boundary conditions at the common-semi-permeable membrane. They adopted the method used by Nunge and Gill (1966) for their analysis of laminar flow in heat exchangers. The essential difference was that while Nunge and Gill had solved the appropriate Sturm-Liouville problem numerically, Cooney *et al.* were able to solve their problem in terms of Kummer's function.

We solve the conjugate boundary value problem of mass transfer in dialysers by applying Galerkin's method described by Mikhlin (1963) and Kantrovich and Krylov (1958), and recently used by Kapur (1980a, b) for solving the mathematically similar problem of diffusion of oxygen in living tissues. The problem is solved under essentially the same assumptions as those of Cooney *et al.* except for the following differences:

(i) We have solved the problem for the circular duct dialyser while Cooney *et al.* had solved it for the flat plate dialyser. This is of some importance because while our method can be easily modified for the flat plate dialyser and we have done it elsewhere (Kapur 1980c), the extension of their method to the circular duct dialyser may lead to a solution in terms of slightly different Kummer's functions.

(ii) They assumed plug flow in the dialyzate region though they noted that this assumption is inconsistent with their other assumption that the effective diffusivity of transverse mass transfer of the dialyzate is essentially equal to the molecular diffusivity. We allow for any velocity profile in the dialyzate region and even allow for flow of dialyzate in a direction opposite to that of blood.

(iii) They assumed blood to be homogeneous and Newtonian. We can consider it non-Newtonian since our method of solution is valid for any velocity profile in the blood region.

(iv) They assumed diffusivities in the two regions to be the same. We allow them to be different.

(v) They neglected axial diffusion and justified it by citing large Pecelets numbers, but there may arise cases where axial diffusion may be important. We show that our method can easily be extended to cover this case.

(vi) Since their solution involved physical parameters, they could not draw conclusions about the convergence of the series solution obtained, though they found the series obtained to be rapidly convergent for the parameters studied by them. In our case, the completeness of the set of Galerkin's functions will ensure the convergence, though the rapidity of convergence will still depend on the values of the parameters considered.

2. The Basic Equations and Boundary Conditions

Let $C_1(r, z)$ and $C_2(r, z)$ be the urea concentrations in the blood and dialyzate regions respectively and let D_1, D_2 be the diffusivities of urea in blood and dialyzate respectively. Let $v_1(r)$, $v_2(r)$ give the velocity distributions in the two regions determined from fluid motion equations and depending on whether the fluids are

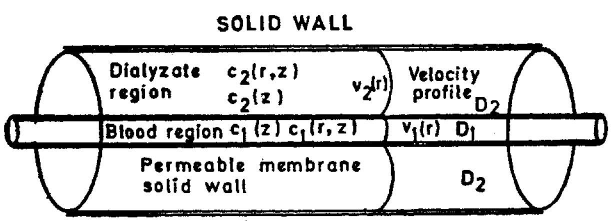

Fig. 1. The Dialyser.

considered Newtonian and when the fluids are Newtonian, depending on their viscosity coefficients, pressure gradients and the radii r_1, r_2 of the inner and outer tubes. In the steady case, the diffusion-convection equations are:

$$D_1\left(\frac{\partial^2 C_1}{\partial r^2} + \frac{1}{r}\frac{\partial C_1}{\partial r}\right) = v_1(r)\frac{\partial C_1}{\partial z} \qquad \ldots(1)$$

$$D_2\left(\frac{\partial^2 C_2}{\partial r^2} + \frac{1}{r}\frac{\partial C_2}{\partial r}\right) = v_2(r)\frac{\partial C_2}{\partial z}. \qquad ...(2)$$

We have to solve these subject to the boundary condition

$$\frac{\partial C_1}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad ...(3)$$

$$\frac{\partial C_2}{\partial r} = 0 \quad \text{at} \quad r = r_2 \qquad ...(4)$$

$$-D_1\left[\frac{\partial C_1}{\partial r}\right]_{r=r_1} = -D_2\left[\frac{\partial C_2}{\partial r}\right]_{r=r_1} = P\,[C_1(r_1, z) - C_2(r_1, z)]$$
$$= P\,[C_1(z) - C_2(z)] \qquad ...(5)$$

where P is the permeability of the membrane and $C_1(z)$, $C_2(z)$ denote the concentrations of urea in the blood and dialyzate sides of the membrane surface. We also have

$$C_1 = C_{10},\; C_2 = C_{20} \quad \text{at} \quad z = 0. \qquad ...(6)$$

Equation (3) arises from consideration of symmetry or from the fact that concentration in the blood region is maximum on the axis. Equation (4) expresses the fact that there is no flux on $r = r_2$. Equation (5) expresses the fact that the rate of urea transfer from blood to the dialyzate across the membrane is proportional to the product of the permeability P of the membrane and the difference of concentrations on the two sides. Finally eqn. (6) gives the initial values of the concentrations at $z = 0$ on the blood and dialyzate sides respectively.

3. Solution by Galerkin's Method

Defining the Laplace transforms

$$\bar{C}_1(r, s) = \int_0^\infty e^{-sz} C_1(r, z)\, dz \qquad ...(7)$$

$$\bar{C}_2(r, s) = \int_0^\infty e^{-sz} C_2(r, z)\, dz \qquad ...(8)$$

and taking the Laplace transform of (1) and (2) and using (5), we get

$$D_1\left(\frac{\partial^2 \bar{C}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{C}_1}{\partial r}\right) = v_1(r)\,[s\bar{C}_1 - C_{10}] \qquad ...(9)$$

$$D_2\left(\frac{\partial^2 \bar{C}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{C}_2}{\partial r}\right) = v_2(r)\,[s\bar{C}_2 - C_{20}]. \qquad ...(10)$$

The boundary condition (3), (4) and (5) are transformed to

$$\frac{\partial \bar{C}_1}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad ...(11)$$

$$\frac{\partial \bar{C}_2}{\partial r} = 0 \quad \text{at} \quad r = r_2 \qquad \text{...(12)}$$

$$-D_1 \left[\frac{\partial \bar{C}_1}{\partial r}\right]_{r=r_1} = -D_2 \left[\frac{\partial \bar{C}_2}{\partial r}\right]_{r=r_1} = [\bar{C}_1(s) - \bar{C}_2(s)] \qquad \text{...(13)}$$

where

$$\bar{C}_1(s) = \bar{C}_1(r_1, s), \bar{C}_2(s) = \bar{C}_2(r_1, s) \qquad \text{...(14)}$$

are the values of the two Laplace transforms at $r = r_1$. We now try the solutions

$$\bar{C}_1(r, s) - \bar{C}_2(s) = \sum_{i=1}^{n} (a_i + b_i r^{2i}) f_i(s) = \sum_{i=1}^{n} h_i(r) f_i(s) \qquad \text{...(15)}$$

$$\bar{C}_2(r, s) - \bar{C}_1(s) = \sum_{i=1}^{n} [A_i + B_i(r - r_1)^{2i}] F_i(s) = \sum_{i=1}^{n} H_i(r) F_i(s) \qquad \text{...(16)}$$

where the $(2n + 2)$ functions

$$f_1(s), f_2(s), ..., f_n(s); F_1(s), F_2(s), ..., F_n(s); \bar{C}_1(s), \bar{C}_2(s) \qquad \text{...(17)}$$

and the $4n$ constants

$$a_1, a_2, ..., a_n; b_1, b_2, ..., b_n; A_1, A_2, ..., A_n; B_1, B_2, ..., B_n \qquad \text{...(18)}$$

have to be determined so that (15) and (16) may satisfy, if possible, the differential eqns. (9) and (10) and the boundary conditions (11), (12) and (13). Functions given in (15) and (16) satisfy the boundary conditions (11) and (12) automatically. The boundary conditions (13) give

$$\left.\begin{aligned} -D_1 \sum_{i=1}^{n} 2ib_i r_1^{2i-1} f_i(s) &= -D_2 \sum_{i=1}^{n} 2iB_i(r_1 - r_2)^{2i-1} F_i(s) \\ = P \sum_{i=1}^{n} (a_i + b_i r_1^{2i}) f_i(s) &= -P \sum_{i=1}^{n} [A_i + B_i(r_1 - r_2)^{2i}] F_i(s). \end{aligned}\right\} \qquad \text{...(19)}$$

Equations (19) are satisfied if we put

$$\left.\begin{aligned} -2ib_i D_1 r_1^{2i-1} = C, &\quad -2iB_i D_2(r_1 - r_2)^{2i-1} = C \\ P(a_i + b_i r_1^{2i}) = C, &\quad -P[A_i + B_i(r_1 - r_2)^{2i}] = C \end{aligned}\right\} \qquad \text{...(20)}$$

and

$$\sum_{i=1}^{n} f_i(s) = \sum_{i=1}^{n} F_i(s) = Q(s), \quad \text{(say)}. \qquad \text{...(21)}$$

Putting $r = r_1$ in (15) and (16) and using (14), we get

$$\bar{C}_1(r_1, s) - \bar{C}_2(s) = \bar{C}_1(s) - \bar{C}_2(s) = \sum_{i=1}^{n} (a_i + b_i r_1^{2i}) f_i(s) \qquad ...(22)$$

$$\bar{C}_2(r_1, s) - \bar{C}_1(s) = \bar{C}_2(s) - \bar{C}_1(s) = \sum_{i=1}^{n} [A_i + B_i(r_1 - r_2)^{2i}] F_i(s). \qquad ...(23)$$

Using (20) and (21), these give

$$\bar{C}_1(s) - \bar{C}_2(s) = \frac{C}{P} \sum_{i=1}^{n} f_i(s) = \frac{C}{P} Q(s) \qquad ...(24)$$

$$\bar{C}_2(s) - \bar{C}_2(s) = -\frac{C}{P} \sum_{i=1}^{n} F_i(s) = -\frac{C}{P} Q(s). \qquad ...(25)$$

Both these equations give

$$L[C_1(z) - C_2(s)] = \frac{C}{P} Q(s). \qquad ...(26)$$

We shall find later that our solution will give all the concerned Laplace transforms as rational functions of s so that we assume

$$Q(s) = K/s^m \qquad ...(27)$$

giving

$$\frac{P}{C} [C_1(z) - C_2(z)] = L^{-1} [Q(s)] = L^{-1} \left[\frac{K}{s^m} \right] = \frac{K}{(m-1)!} \cdot z^{m-1}. \qquad ...(28)$$

However

$$C_1(0) - C_2(0) = C_{10} - C_{20} \qquad ...(29)$$

so that we choose

$$m = 1,\ K = \frac{P}{C} [C_{10} - C_{20}]. \qquad ...(30)$$

Thus (21), (27) and (30) give

$$\sum_{i=1}^{n} f_i(s) = \sum_{i=1}^{n} F_i(s) = \frac{P}{C} \frac{C_{10} - C_{20}}{s}. \qquad ...(31)$$

It is obvious that there is no loss of generality if we give any specific value to C. Let us choose $C = P$ so that we satisfy (19) by specifying

$$a_i = 1 + \frac{1}{2i}\frac{Pr_1}{D_1}, \quad A_i = -1 - \frac{1}{2i}\frac{P}{D_2}(r_2 - r_1) \qquad \text{...(32)}$$

$$b_i = -\frac{1}{2i}\frac{Pr_1}{D_1}\frac{1}{r_1^{2i}}, \quad B_i = \frac{1}{2i}\frac{P(r_2 - r_1)}{D_2}\frac{1}{(r_2 - r_1)^{2i}} \qquad \text{...(33)}$$

$$h_i(r_1) = a_i + b_i r_1^{2i} = 1, \; H_i(r_1) = A_i + B_i(r_2 - r_1)^{2i} = -1 \qquad \text{...(34)}$$

for all values of $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} f_i(s) = \frac{C_{10} - C_{20}}{s}, \quad \sum_{i=1}^{n} F_i(s) = \frac{C_{10} - C_{20}}{s}. \qquad \text{...(35)}$$

From (31) - (33), we find that

$$a_i > 0, \quad A_i < 0; \quad i = 1, 2, \ldots, n \qquad \text{...(36)}$$

$$b_i < 0, \quad B_i > 0; \quad i = 1, 2, \ldots, n \qquad \text{...(37)}$$

$$h_i(r_1) > 0, \quad H_i(r_1) < 0; \quad i = 1, 2, \ldots, n \qquad \text{...(38)}$$

$$h_i(r) = a_i + b_i r^{2i} = a_i + b_i r_1^{2i} + b_i(r^{2i} - r_1^{2i})$$

$$= 1 + \frac{P}{D_1}\frac{r_1}{2i}\left[1 - \left(\frac{r}{r_1}\right)^{2i}\right] > 0 \quad \text{when} \quad r < r_1 \qquad \text{...(39)}$$

$$H_i(r) = A_i + B_i(r - r_2)^{2i} = A_i + B_i(r_2 - r_1)^{2i}$$

$$+ B_i(r_2 - r)^{2i} - B_i(r_2 - r_1)^{2i}$$

$$= -1 - \frac{P}{D_2}\frac{1}{2i}(r_2 - r_1)\left[1 - \left(\frac{r_2 - r}{r_2 - r_1}\right)^{2i}\right]$$

$$< 0 \quad \text{when} \quad r_1 < r < r_2. \qquad \text{...(40)}$$

Thus by satisfying the boundary conditions, we have determined all the $4n$ constants a_i, b_i, A_i, B_i and one relation each among the functions $f_i(s)$ and $F_i(s)$. We had to determine $(2n + 2)$ functions and as such we have to determine the $2n$ more relations to determine these functions. We determine these relations and functions in the next section.

4. Determination of $f_i(s)$, $F_i(s)$, $\bar{C}_1(s)$, $\bar{C}_2(s)$, $C_1(z)$, $C_2(z)$, $C_1(r, z)$, $C_2(r, z)$

The functions $\bar{C}_1(r, s)$, $\bar{C}_2(r, s)$ defined by (15) and (16) do not satisfy the differential eqns. (9) and (10) though these satisfy the boundary conditions (11), (12) and (13) exactly. We would like to choose the unknown functions in such a way that (15) and (16) satisfy (9) and (10) as closely as possible.

According to Galerkin's method, we first find residuals $R_1(r, s)$, $R_2(r, s)$ which remain when we substitute from (15) and (16) in (9) and (10) and then choose the unknown functions in such a way that these residuals are orthogonal to $h_j(r)$, $j = 1, 2, \ldots, n$; and $H_j(r)$, $j = 1, 2, \ldots, n$ respectively. The $2n$ orthogonality relations would give us the $2n$ additional relations required to determine all the unknown functions. We find from (9) and (15)

$$R_1(r, s) = D_1 \sum_{i=1}^{n} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial h_i}{\partial r} \right) f_i(s)$$

$$= v_1(r) \left\{ s \left[\bar{C}_2(s) + \sum_{i=1}^{n} h_i(r) f_i(s) \right] - C_{10} \right\}. \quad \ldots(41)$$

Similarly from (10) and (16)

$$R_2(r, s) = D_2 \sum_{i=1}^{n} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H_i}{\partial r} \right) F_i(s) - v_2(r) \left\{ s \left[\bar{C}_1(s) \right.\right.$$

$$\left.\left. + \sum_{i=1}^{n} H_i(r) F_i(s) \right] - C_{10} \right\}. \quad \ldots(42)$$

The orthogonality relations give

$$\int_0^{r_1} R_1(r, s)\, h_j(r)\, r\, dr = 0; \quad \int_{r_1}^{r_2} R_2(r, s)\, H_j(r)\, (r - r_2)\, dr = 0;$$

$$j = 1, 2, \ldots, n. \quad \ldots(43)$$

These give

$$\sum_{i=1}^{n} (a_{ij} + s b_{ij})\, F_i(s) = p_j \left[s\bar{C}_2(s) - C_{10} \right]; \ j = 1, 2, \ldots, n \quad \ldots(44)$$

$$\sum_{i=1}^{n} (A_{ij} + s B_{ij})\, F_i(s) = P_j \left[s\bar{C}_1(s) - C_{20} \right]; \ j = 1, 2, \ldots, n \quad \ldots(45)$$

where

$$a_{ij} = -D_1 \int_0^{r_1} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial h_i}{\partial r} \right) h_j(r)\, r\, dr \quad \ldots(46)$$

$$b_{ij} = \int_0^{r_1} v_1(r)\, h_i(r)\, h_j(r)\, r\, dr \quad \ldots(47)$$

$$p_j = -\int_0^{r_1} v_1(r)\, h_j(r)\, r\, dr \qquad ...(48)$$

$$A_{ij} = -D_2 \int_{r_1}^{r_2} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_i}{\partial r}\right) H_j(r)\,(r - r_2)\, dr \qquad ...(49)$$

$$B_{ij} = \int_{r_1}^{r_2} v_2(r)\, H_i(r)\, H_j(r)\,(r - r_2)\, ds \qquad ...(50)$$

$$P_j = -\int_{r_1}^{r_2} v_2(r)\, H_j(r)\,(r - r_2)\, dr. \qquad ...(51)$$

Now

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h_i}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(2ib_i r^{2i}) = 4i^2 b_i r^{2i-2} < 0 \qquad ...(52)$$

$$a_{ij} = -4i^2 b_i D_1 \int_0^{r_1} r^{2i-1}(a_j + b_j r^{2j})\, dr$$

$$= 2i\frac{P}{D_1}\frac{r_1}{r_1^{2i}} D_1\left[\left(1 + \frac{Sh_w}{2j}\right)\left(\frac{r_1^{2i}}{2i} - \frac{Sh_w}{2j}\frac{r_1^{2i}}{2i + 2j}\right]\right.$$

$$= D_1 Sh_w\left[1 + \frac{Sh_w}{2i + 2j}\right] \qquad ...(53)$$

where

$$Sh_w = \frac{Pr_1}{D_1} \qquad ...(54)$$

is the dimensionless Sherwood number at the wall corresponding to diffusivity D_1. We can also define another Sherwood number

$$Sh'_w = \frac{P(r_2 - r_1)}{D_2} \qquad ...(55)$$

corresponding to diffusivity D_2 and m terms of these, (32) and (33) become

$$a_j = 1 + \frac{Sh_w}{2i}, \quad A_i = -1 - \frac{1}{2i}Sh'_w \qquad ...(56)$$

$$b_i = -\frac{1}{2i}Sh_w\frac{1}{r_1^{2i}}, \quad B_i = \frac{1}{2i}\frac{Sh'_w}{(r_2 - r_1)^{2i}}. \qquad ...(57)$$

We easily see that (a_{ij}), (b_{ij}) are symmetric matrices with all positive elements, (B_{ij}) is symmetric and has all negative elements if $v_2(r) > 0$ i.e. if the dialyzate flow is in same direction as blood flow and (B_{ij}) is symmetric and has all positive elements if $v_2(r) < 0$ i.e. if the dialyzate flow is in the opposite direction to that of blood flow. All p_j's are negative and all P_j's are negative if $v_2(r) > 0$ and are positive if $v_2(r) < 0$.

Substituting

$$g_i(s) = \frac{f_i(s)}{s\bar{C}_2(s) - C_{10}}, \quad G_i(s) = \frac{F_i(s)}{s\bar{C}_1(s) - C_{20}} \qquad \text{...(58)}$$

(44) and (45) give

$$\sum_{i=1}^{n} (a_{ij} + sb_{ij})\, g_i(s) = p_j; \; j = 1, 2, \ldots, n \qquad \text{...(59)}$$

$$\sum_{i=1}^{n} (A_{ij} + sB_{ij})\, G_i(s) = P_j; \; j = 1, 2, \ldots, n. \qquad \text{...(60)}$$

From the system of linear eqns. (59) and (60), we can easily solve for $g_i(s)$ and $G_i(s)$ and find that each is an $R(n-1, n)$ function i.e. each is a rational function of s whose numerator is a polynomial of $(n-1)$th degree and whose denominator is a polynomial of nth degree. From (35) and (58)

$$s\bar{C}_2(s) - C_{10} = \frac{C_{10} - C_{20}}{s \sum_{i=1}^{n} g_i(s)}, \quad s\bar{C}_1(s) - C_{20} = \frac{C_{10} - C_{20}}{s \sum_{i=1}^{n} G_i(s)}. \qquad \text{...(61)}$$

Now $\sum_{i=1}^{n} g_i(s)$ and $\sum_{i=1}^{n} G_i(s)$ are $R(n-1, n)$ functions and

$$\bar{C}_2(s) = \frac{C_{10}}{s} + \frac{C_{10} - C_{20}}{s^2} \; [\text{an } R(n, n-1) \text{ function}] \qquad \text{...(62)}$$

$$\bar{C}_1(s) = \frac{C_{20}}{s} + \frac{C_{10} - C_{20}}{s^2} \; [\text{an } R(n, n-1) \text{ function}] \qquad \text{...(63)}$$

so that $\bar{C}_1(s)$, $\bar{C}_2(s)$ are $R(n, n+1)$ functions. Finally

$$f_i(s) = \frac{C_{10} - C_{20}}{s} \frac{g_i(s)}{\sum_{i=1}^{n} g_i(s)}, \quad F_i(s) = \frac{C_{10} - C_{20}}{s} \frac{G_i(s)}{\sum_{i=1}^{n} G_i(s)} \qquad \text{...(64)}$$

so that both $f_i(s)$ and $F_i(s)$ are $R(n-1, n)$ functions. We now easily get

$$C_1(z) = L^{-1}[\bar{C}_1(s)], \quad C_2(z) = L^{-1}[\bar{C}_2(s)] \qquad \text{...(65)}$$

$$C_1(r, z) - C_2(z) = \sum_{i=1}^{n} (a_i + b_i r^{2i})\, L^{-1}[f_i(s)] \qquad \text{...(66)}$$

$$C_2(r, z) - C_1(z) = \sum_{i=1}^{n} (A_i + B_i(r - r_2)^{2i}] L^{-1} [F_i(s)] \quad ...(67)$$

so that $C_1(z)$, $C_2(z)$, $C_1(r, z)$, $C_2(r, z)$ are all known for any specified value of n.

Since (a_{ij}) and (b_{ij}) are positive symmetric matrices, the zeros of the determinant $| a_{ij} + b_{ij}s |$ are all real and negative. As such from (59) we get that $g_i(s)$ is of the form

$$g_i(s) = \frac{A_{i1}}{s + r_1} + \frac{A_{i2}}{s + r_2} + \dots + \frac{A_{in}}{s + r_n} \quad ...(68)$$

$$\sum_{i=1}^{n} g_i(s) = \frac{A_1}{s + r_1} + \frac{A_2}{s + r_2} + \dots + \frac{A_n}{s + r_n};\ A_j = \sum_{i=1}^{n} A_{ij} \quad ...(69)$$

$$f_i(s) = \frac{C_{10} - C_{20}}{s} \frac{\sum_{j=1}^{n} A_{ij}(s + r_1) \dots (s + r_{j-1})(s + r_{j+1}) \dots (s + r_n)}{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}(s + r_1) \dots (s + r_{j-1})(s + r_{j+1}) \dots (s + r_n)} . \quad ...(70)$$

5. Consideration of Axial Diffusion

When axial diffusion is considered in both regions, the basic eqns. (1) and (2) and their Laplace transforms (9) and (10) are modified to

$$D_1 \left(\frac{\partial^2 C_1}{\partial r^2} + \frac{1}{r} \frac{\partial C_1}{\partial r} \right) + D_1' \frac{\partial^2 C_1}{\partial z^2} = v_1(r) \frac{\partial C_1}{\partial z} \quad ...(71)$$

$$D_2 \left(\frac{\partial^2 C_2}{\partial r^2} + \frac{1}{r} \frac{\partial C_2}{\partial r} \right) + D_2' \frac{\partial^2 C_2}{\partial z^2} = v_2(r) \frac{\partial C_2}{\partial z} \quad ...(72)$$

$$D_1 \left(\frac{\partial^2 \bar{C}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{C}_1}{\partial r} \right) + D_1' (s^2 \bar{C}_1 - sC_{10}) = v_1(r) (s\bar{C}_1 - C_{10}) \quad ...(73)$$

$$D_2 \left(\frac{\partial^2 \bar{C}_2}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{C}_2}{\partial r} \right) + D_2' (s^2 \bar{C}_2 - sC_{20}) = v_2(s) (s\bar{C}_2 - C_{20}) \quad ...(74)$$

where we have assumed the additional boundary conditions

$$\frac{\partial C_1}{\partial z} = 0, \quad \frac{\partial C_2}{\partial z} = 0 \quad \text{at} \quad z = 0. \quad ...(75)$$

If these fluxes are not zero, these have to be prescribed as known functions $f_1(r)$, $f_2(r)$. In that case (73), (74) become

$$D_1 \left(\frac{\partial^2 \bar{C}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{C}_1}{\partial r} \right) + D_1' (s^2 \bar{C}_1 - sC_{10} - f_1(r)) = v_1(r) (s\bar{C}_1 - C_{10}) \quad ...(76)$$

$$D_2\left(\frac{\partial^2 \bar{C}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{C}_2}{\partial r}\right) + D_2'\,(s^2\bar{C}_2 - sC_{20} - f_2(r)) = v_2(r)\,(s\bar{C}_2 - C_{20}). \quad \ldots(77)$$

In our method, these equations can be handled without any special difficulty. However for simplicity, we shall assume (75) to hold. In this case the residuals given by (41) and (42) will be modified and the modified residuals $\bar{R}_1(r, s)$, $\bar{R}_2(r, s)$ will be obtained from $R_1(r, s)$, $R_2(r, s)$ by replacing $v_1(r)$, $v_2(r)$ by $v_2(r) - D_1' s$ and $v_2(r) - D_2' s$ respectively.

Using (47), (48), (50), (51) we find that b_{ij}, B_{ij}, p_j, P_j will be replaced by

$$b_{ij} - sC_{ij},\ B_{ij} - sC_{ij},\ p_j + sq_j,\ P_j + sQ_j,$$

where

$$C_{ij} = D_1' \int_0^{r_1} h_i(r)\,h_j(r)\,r\,dr \quad \ldots(78)$$

$$C_{ij} = D_2' \int_{r_1}^{r_2} H_i(r)\,H_j(r)\,(r - r_2)\,dr \quad \ldots(79)$$

$$q_j = D_1' \int_0^{r_1} h_j(r)\,r\,dr \quad \ldots(80)$$

$$Q_u = D_2' \int_{r_1}^{r_2} H_j(r)\,(r - r_2)\,dr \quad \ldots(81)$$

so that eqns. (44) and (45) are modified to

$$\sum_{i=1}^{n} (a_{ij} + sb_{ij} - s^2C_{ij})\,f_i(s) = (p_j + sq_j)\,(s\bar{C}_2(s) - C_{10}),\ j = 1, 2, \ldots, n \quad \ldots(82)$$

$$\sum_{i=1}^{n} (A_{ij} + sB_{ij} - s^2C_{ij})\,F_i(s) = (P_j + sQ_j)\,(s\bar{C}_1(s) - C_{20}),\quad j = 1, 2, \ldots, n \quad \ldots(83)$$

which together with (34) can be solved for $f_i(s)$, $F_i(s)$, $\bar{C}_1(s)$, $\bar{C}_2(s)$. In fact eqn. (82) together with

$$\sum_{i=1}^{n} f_i(s) = \frac{C_{10} - C_{20}}{s} \quad \ldots(84)$$

determine $f_1(s), f_2(s), \ldots, f_n(s)$ and $\bar{C}_2(s)$ and eqn. (83) together with

$$\sum_{i=1}^{n} F_i(s) = \frac{C_{10} - C_{20}}{s} \quad \text{...(85)}$$

determine $F_1(s)$, $F_2(s)$, ..., $F_n(s)$ and $\bar{C}_1(s)$.

In case $v_1(r)$, $v_2(r)$ can be replaced by the average velocities $\bar{v}_1$, $\bar{v}_2$, eqns. (82) and (83) are simplified to

$$\sum_{i=1}^{n} \left(a_{ij} + s \frac{\bar{v}_1}{D_1'} C_{ij} - s^2 C_{ij} \right) f_i(s) = q_j \left(s - \frac{\bar{v}_1}{D_1'} \right) [s\bar{C}_2(s) - C_{10}] \quad \text{...(86)}$$

$$\sum_{i=1}^{n} \left(A_{ij} + s \frac{\bar{v}_2}{D_2'} C_{ij} - s^2 C_{ij} \right) F_i(s) = Q_j \left(s - \frac{\bar{v}_2}{D_2'} \right) (s\bar{C}_1(s) - C_{10}). \quad \text{...(87)}$$

By substituting

$$k_i(s) = \frac{f_i(s)}{\left(s - \frac{\bar{v}_1}{D_1'} \right) (s\bar{C}_2(s) - C_{10})} \quad \text{...(88)}$$

$$K_i(s) = \frac{F_i(s)}{\left(s - \frac{\bar{v}_2}{D_2'} \right) (s\bar{C}_1(s) - C_{10})} \quad \text{...(89)}$$

we obtain two systems of linear equations to determine $k_i(s)$ and $K_i(s)$ and then using (34), we can find $f_i(s)$, $F_i(s)$. As before $f_i(s)$, $F_i(s)$, $\bar{C}_1(s)$, $\bar{C}_2(s)$ come out to be rational functions and by inverting these we can get $C_1(r, z)$, $C_2(r, z)$, $C_1(z)$ and $C_2(z)$.

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