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THE MAXIMAL k-FULL DIVISOR OF AN INTEGER

D. SURYANARAYANA AND P. SUBRAHMANYAM*

Department of Mathematics, Andhra University, Waltair 530003

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Let k be a fixed integer $\geqslant 2$. A positive integer n is called k-full if $p^k \mid n$ for every prime factor p of n. The integer 1 is also considered to be k-full. Let $M_k(n)$ denote the maximal k-full divisor of n. In this paper we obtain an asymptotic formula for $\sum_{n \leqslant x} M_k(n)$ and deduce an asymptotic formula for $\sum_{n \leqslant x} \frac{1}{\gamma_k^*(n)}$, where $\gamma_k^*(n)$ denotes the maximal unitary k-free divisor of n.

1. INTRODUCTION

Let k be a fixed integer $\geqslant 2$. A positive integer n is called k-full if in the cannonical representation of n into prime powers each exponent is $\geqslant k$ and a positive integer n is called k-free if in the cannonical representation of n into prime powers each exponent is $< k$. The integer 1 is considered to be both k-full and k-free. Let L_k and Q_k denote the sets of k-full and k-free integers respectively. Further let $l_k(n)$ denote the characteristic function of the set L_k i.e. $l_k(n) = 1$ or 0 according as $n \in L_k$ or $n \notin L_k$, we have $l_2(n) = l(n)$, the characteristic function of the square-full integers, and let $q_k(n)$ denote the characteristic function of the k-free integers, we have $q_2(n) = q(n)$, the characteristic function of the square-free integers. We note that both $l_k(n)$ and $q_k(n)$ are multiplicative for any $k \geqslant 2$. For real $x \geqslant 1$, let $L_k(x; n)$ be the number of integers $\leqslant x$, cotained in L_k and are prime to n. In case $n = 1$, clearly $L_k(x; n) = L_k(x)$, the number of k-full integers $\leqslant x$. An elementary estimate for $L_k(x)$ was obtained by Erdös and Szekeres (1934). Later, Bateman and Grosswald (1958) obtained better estimates for $L_k(x)$, using analytic methods. Recently, Cohen and Davis (1970) established certain weaker estimates for $L_k(x)$ by elementary methods. For the work done on the asymptotic for $L_k(x)$, we refer to the bibliography by Cohen (1963), Cohen and Davis (1970), Ivić (1978) and Ivić and Shin (1980). Also, we refer to the work of Suryanarayana and Rao (1973) and Suryanarayana (1979) for the best known results on the O-estimates of the error terms in the asymptotic formula for $L_2(x)$.

Let $M_k(n)$ denote the maximal k-full divisor of n. In this paper we obtain an estimate for the summatary function of $M_k(n)$ by elementary methods. For this

*Sriram Junior College, Sriramnagar, Garividi, Viziayanagaram Distt. (A.P.).

purpose we need an asymptotic formula for $L_k(x; n)$ which we obtained by adopting the method of Cohen and Davis (1970). Further, we deduce an asymptotic formula for the summatary function of $\frac{1}{\gamma_k^*(n)}$, where $\gamma_k^*(n)$ denotes the maximal unitary k-free divisor of n. By a unitary divisor of n, we mean a divisor d of n such that $(d, n/d) = 1$.

In section 2 we state some lemmas established by Cohen and Davis (1970) and prove some more lemmas which we need in our present discussion. In section 3 we prove the main results.

2. Preliminaries

Let $\varphi(x, n)$ denote the number of positive integers $\leqslant x$ co-prime to n and let $\sigma^*(s, n)$ be the sum of the sth powers of the square-free divisors of n. Clearly $\sigma^*(0, n) = \theta(n)$, the number of square-free divisors of n. Let $\varphi_s(n)$ be the function defined by

$$\varphi_s(n) = \sum_{d \mid n} \mu(d) \left(\frac{n}{d}\right)^s \text{ for any real } s > 0 \qquad \ldots(2.1)$$

where $\mu(n)$ denotes the Möbius function.

Clearly $\varphi_1(n) = \varphi(n)$ where $\varphi(n)$ is the Eulers totient function. The function $\varphi_s(n)$ has the representation

$$\varphi_s(n) = n^s \prod_{p \mid n} \left(1 - \frac{1}{p^s}\right). \qquad \ldots(2.2)$$

Remark 2.1 : We note that $\varphi_s(n) \leqslant n^s$ for $s \geqslant 1$ and $1/\varphi_s(n) \leqslant \zeta(s)/n^s$ for $s > 1$, where $\zeta(s)$ denotes the Riemann Zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Lemma 2.1 (cf. Cohen 1964, eqn. (4)) — For $0 \leqslant \alpha < 1$,

$$\varphi(x, n) = \frac{x\varphi(n)}{n} + O(x^\alpha \sigma^*(-\alpha, n)), \qquad \ldots(2.3)$$

uniformly in both x and n.

Lemma 2.2 (cf. Cohen and Davis 1970, Lemma 5) — If $s > 0$, $s \neq 1$, $x \geqslant 1$, then for $0 \leqslant \alpha < 1$,

$$N_s(x, n) \equiv \sum_{\substack{m \leqslant x \\ (m, n)=1}} \frac{1}{m^s} = \frac{\zeta(s)\,\varphi_s(n)}{n^s} - \frac{\varphi(n)}{n(s-1)\,x^{s-1}} + O(x^{\alpha-s}\sigma^*(-\alpha, n)), \qquad \ldots(2.4)$$

uniformly both in x and n; where for $0 < s < 1$, $\zeta(s)$ is defined by

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty (t - [t])\, t^{-s-1} dt. \qquad ...(2.5)$$

Lemma 2.3 (cf. Cohen and Davis 1970, Lemma 3) — If $0 < \alpha < \frac{k-1}{k+2}$, then

$$\sum_{\substack{n \leqslant x \\ n \in L_{k+2}}} \frac{\sigma^*(-\alpha, n)}{n^{(\alpha+1)/(2k+1)}} = O(x^{1/(k+2)-(\alpha+1)/(2k+1)}). \qquad ...(2.6)$$

Lemma 2.4 — For any $\alpha > 0$,

$$\sum_{\substack{n \leqslant x \\ n \in L_{k+2}}} \frac{\sigma^*(-\alpha, n)}{n^{1/(2k+2)}} = O(x^{k/(k+2)(2k+2)}). \qquad ...(2.7)$$

PROOF: Equation (2.7) can be proved following the same lines given by Cohen and Davis (1970, Lemma 3) in their proof of (2.6) above.

Lemma 2.5 (cf. Cohen and Davis 1970, Lemma 2) — (a) For $0 < s < 1/k$

$$\sum_{n \leqslant x} \frac{l_k(n)}{n^s} = O(x^{(1-sk)/k}), \quad x \geqslant 1. \qquad ...(2.8)$$

(b) For $s = 1/k$

$$\sum_{n \leqslant x} \frac{l_k(n)}{n^s} = O(\log x), \quad x \geqslant 2. \qquad ...(2.9)$$

(c) For $s > 1/k$

$$\sum_{n > x} \frac{l_k(n)}{n^s} = O(x^{(1-sk)/k}), \quad x \geqslant 1. \qquad ...(2.10)$$

Let B_k denote the set of those positive integers all of whose prime divisors have multiplicity in the interval $[k + 2, 2k)$.

Remark 2.2 : (a) $B_2 = \{1\}$

(b) $B_k \subseteq L_{k+2}$.

Lemma 2.6 (cf. Cohen and Davis 1970, Lemma 8) — For $k \geqslant 2$

$$l_k(n) = \sum_{d^k e^{2k+2} f^{k+1} h = n} \mu(e) \qquad ...(2.11)$$

where the summation is over integers d, e, f and h such that $h \in B_k$ and $(e, h) = (f, h) = 1$.

Lemma 2.7 — If $1 \leqslant a < b$, $0 \leqslant \alpha < a/b$, then for $x \geqslant 1$,

$$T_{a,b}^{m,h}(x) = \sum_{\substack{d^a f^b \leqslant x \\ (d,m)=(f,h)=1}} 1 = A_{m,h}(a, b)\, x^{1/a} + B_{m,h}(a, b)\, x^{1/b}$$

$$+ O(x^{(\alpha+1)/(a+b)} \rho_\alpha(h, m)) + O(x^{2\alpha/(a+b)} \sigma^*(-\alpha, h)\, \sigma^*(-\alpha, m)), \qquad ...(2.12)$$

uniformly where

$$\rho_\alpha(h, m) = \max(\sigma^*(-\alpha, h), \sigma^*(-\alpha, m)) \qquad ...(2.13)$$

$$A_{m,h}(a, b) = \frac{\zeta(b/a)\, \varphi(m)\, \varphi_{b/a}(h)}{m h^{b/a}} \qquad ...(2.14)$$

$$B_{m,h}(a, b) = \frac{\zeta(a/b)\, \varphi(h)\, \varphi_{a/b}(m)}{h m^{a/b}} \qquad ...(2.15)$$

PROOF : We have

$$T_{a,b}^{m,h}(x) = \sum_{\substack{d^a f^b \leqslant x \\ (d,m)=(f,h)=1}} 1 = \sum_{\substack{d \leqslant x^{1/(a+b)} \\ d^a f^b \leqslant x \\ (d,m)=(f,h)=1}} 1$$

$$+ \sum_{\substack{f \leqslant x^{1/(a+b)} \\ d^a f^b \leqslant x \\ (d,m)=(f,h)=1}} 1 - \sum_{\substack{d, f \leqslant x^{1/(a+b)} \\ d^a f^b \leqslant x \\ (d,m)=(f,h)=1}} 1$$

$$= S_1 + S_2 - S_3, \text{ say.} \qquad ...(2.16)$$

We have by Lemmas 2.1 and 2.2 and the fact that $a\alpha/b < 1$,

$$S_1 = \sum_{\substack{d \leqslant x^{1/(a+b)} \\ (d,m)=1}} \sum_{\substack{f \leqslant (x^{1/b})/(d^{a/b}) \\ (f,h)=1}} = \sum_{\substack{d \leqslant x^{1/(a+b)} \\ (d,m)=1}} \varphi\left(\frac{x^{1/b}}{d^{a/b}}, h\right)$$

$$= \sum_{\substack{d \leqslant x^{1/(a+b)} \\ (d,m)=1}} \left\{\frac{x^{1/b}}{d^{a/b}} \frac{\varphi(h)}{h} + O\left(\frac{x^{\alpha/b}}{d^{a\alpha/b}}\, \sigma^*(-\alpha, h)\right)\right\}$$

(*equation continued on p.* 179)

$$= x^{1/b} \frac{\varphi(h)}{h} \sum_{\substack{d \leqslant x^{1/(a+b)} \\ (d,m)=1}} \frac{1}{d^{a/b}} + O\Big(x^{\alpha/b} \sigma^*(-\alpha, h) \sum_{\substack{d \leqslant x^{1/(a+b)} \\ (d,m)=1}} \frac{1}{d^{a\alpha/b}}\Big)$$

$$= x^{1/b} \frac{\varphi(h)}{h} N_{a/b}(x^{1/(a+b)}, m) + O(x^{\alpha/b} \sigma^*(-\alpha, h)\,(x^{1/(a+b)})^{(b-a\alpha)/b})$$

$$= x^{1/b} \frac{\varphi(h)}{h} \left\{ \frac{\zeta(a/b)\, \varphi_{a/b}(m)}{m^{a/b}} - \frac{\varphi(m)}{m} \frac{b}{(a-b)} \frac{1}{(x^{1/(a+b)})^{(a-b)/b}} \right.$$

$$\left. + O((x^{1/(a+b)})^{(\alpha-a)/b}\, \sigma^*(-\alpha, m)) \right\} + O(x^{(\alpha+1)/(a+b)} \sigma^*(-\alpha, h))$$

$$= \frac{\zeta(a/b)\, \varphi(h)\, \varphi_{a/b}(m)}{h m^{a/b}} x^{1/b} - \frac{b}{a-b} \frac{\varphi(h)}{h} \frac{\varphi(m)}{m} x^{2/(a+b)}$$

$$+ O(\rho_\alpha(h, m)\, x^{(\alpha+1)/(a+b)}). \qquad \text{...(2.17)}$$

If in S_1, we interchange a and b and further h and m, we get the value of S_2, thus

$$S_2 = \frac{\zeta(b/a)\, \varphi(m)\, \varphi_{b/a}(h)}{m h^{b/a}} x^{1/a} - \frac{a}{(b-a)} \frac{\varphi(m)\, \varphi(h)}{mh} x^{2/(a+b)}$$

$$+ O(\rho_\alpha(h, m)\, x^{(\alpha+1)/(a+b)}). \qquad \text{...(2.18)}$$

Again by Lemma 2.1, we have

$$S_3 = \sum_{\substack{f, d \leqslant x^{1/(a+b)} \\ d^a f^b \leqslant x \\ (d,m)=(f,h)=1}} 1 = \sum_{\substack{d \leqslant x^{1/(a+b)} \\ (d,m)=1}} \sum_{\substack{f \leqslant x^{1/(a+b)} \\ (f,h)=1}} 1 = \varphi(x^{1/(a+b)}, m)$$

$$\times \varphi(x^{1/(a+b)}, h)$$

$$= \left[\frac{\phi(m)}{m} x^{1/(a+b)} + O(x^{\alpha/(a+b)}\, \sigma^*(-\alpha, m))\right] \left[\frac{\varphi(h)}{h} x^{1/(a+b)} \right.$$

$$\left. + O(x^{\alpha/(a+b)} \sigma^*(-\alpha, h))\right]$$

$$= \frac{\varphi(m)\, \varphi(h)}{mh} x^{2/(a+b)} + O(x^{(\alpha+1)/(a+b)} \rho_\alpha(h, m))$$

$$+ O(x^{2\alpha/(a+b)} \sigma^*(-\alpha, m)\, \sigma^*(-\alpha, h). \qquad \text{...(2.19)}$$

Now, Lemma 2.7 follows from (2.16), (2.17), (2.18) and (2.19).

Remark 2.3 : A proof of Lemma 2.7 has been given by Cohen and Davis (1970, Theorem 3). However, second O-term in (2.12) above has been over looked and not mentioned by Cohen and Davis.

Remark 2.4 : For $(h, n) = 1$, it is clear by the multiplicativity of $\varphi_s(n)$ and from (2.14) and (2.15) that,

$$A_{n,hn}(a, b) = \frac{\varphi(n)\,\varphi_{b/a}(n)}{n^{(a+b)/a}} A_{1,h}(a, b) \quad \text{...(2.20)}$$

$$B_{n,hn}(a, b) = \frac{\varphi(n)\,\varphi_{a/b}(n)}{n^{(a+b)/b}} B_{1,h}(a, b) \quad \text{...(2.21)}$$

where

$$A_{1,h}(a, b) = \frac{\zeta(b/a)\,\varphi_{b/a}(h)}{h^{b/a}} \quad \text{...(2.22)}$$

$$B_{1,h}(a, b) = \frac{\zeta(a/b)\,\varphi(h)}{h}. \quad \text{...(2.23)}$$

Remark 2.5 : We note that $A_{1,h}(a, b)$ and $B_{1,h}(a, b)$ are bounded.

Lemma 2.8 — For any real numbers s and j, $s > j + 1$,

$$\sum_{m=1}^{\infty} \frac{q_k(m)\,\varphi(m)\,\varphi_j(m)}{m\varphi_s(m)} = \prod_p \left\{ \frac{(p^{(s-j)k} - 1)(p^j - 1)(p - 1)}{(p^{s-j} - 1)(p^s - 1)\,p^{(s-j)(k-2)+1}} \right\}. \quad \text{...(2.24)}$$

PROOF : By Remark 2.1, it is easy to observe that the series in (2.24) is absolutely convergent and the general term of the series is multiplicative. Hence the series can be expanded into an infinite Euler product (cf. Hardy and Wright 1968, Theorem 286), and (2.24) follows by easy calculation.

Lemma 2.9 — For any real numbers s and j, $s > j + 1$ and for $x \geqslant 2$,

$$\sum_{m \leqslant x} \frac{q_k(m)\,\varphi(m)\,\varphi_j(m)}{m\varphi_s(m)} = \prod_p \left\{ \frac{(p^{(s-j)k} - 1)(p^j - 1)(p - 1)}{(p^{s-j} - 1)(p^s - 1)\,p^{(s-j)(k-2)+1}} \right\} + O(x^{j-s+1}). \quad \text{...(2.25)}$$

PROOF : By Remark 2.1, we have

$$\sum_{m > x} \frac{q_k(m)\,\varphi(m)\,\varphi_j(m)}{m\varphi_s(m)} = O\left(\sum_{m > x} \frac{q_k(m)}{m^{s-j}} \right)$$

$$= O\left(\sum_{m > x} \frac{1}{m^{s-j}} \right) = O(x^{j-s+1}).$$

Now, Lemma 2.9 follows from Lemma 2.8 and the above O-estimate. From Lemma 2.9, we get the following :

For $x \geqslant 2$,

$$\sum_{m \leqslant x} \frac{q_k(m)\,\varphi(m)\,\varphi_{(k+1)/k}(m)}{m\varphi_{(2k+2)/k}(m)} = \alpha_k + O(x^{-1/k}) \quad \text{...(2.26)}$$

and

$$\sum_{m \leqslant x} \frac{q_k(m)\, \varphi(m)\, \varphi_{k/(k+1)}(m)}{m \varphi_2(m)} = \beta_k + O(x^{-1/(k+1)}) \qquad \ldots(2.27)$$

where α_k and β_k are the constants given by

$$\alpha_k = \prod_p \left\{ \frac{(p^{k+1}-1)\,(p^{(k+1)/k}-1)\,(p-1)}{(p^{(k+1)/k}-1)\,(p^{(2k+2)/k}-1)\,(p^{(k+1)(k-2)/k}+1)} \right\} \qquad \ldots(2.28)$$

$$\beta_k = \prod_p \left\{ \frac{(p^{k(k+2)/(k+1)}-1)\,(p^{k/(k+1)}-1)}{(p^{(k+2)/(k+1)}-1)\,(p+1)\,p^{(k^2-4)/(k+1)}+1} \right\}. \qquad \ldots(2.29)$$

Lemma 2.10 — For $x \geqslant 1$ and for fixed positive integer n,

$$\sum_{\substack{h \leqslant x \\ (h,n)=1 \\ h \in B_k}} \frac{A_{1,h}(a,b)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} = \sum_{\substack{h=1 \\ (h,n)=1 \\ h \in B_k}}^{\infty} \frac{A_{1,h}(a,b)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} + O(x^{-2/k(k+2)}). \qquad \ldots(2.30)$$

PROOF: We have by Remarks 2.1 and 2.5

$$\sum_{\substack{h \leqslant x \\ (h,n)=1 \\ h \in B_k}} \frac{A_{1,h}(a,b)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} = \sum_{\substack{h=1 \\ (h,n)=1 \\ h \in B_k}}^{\infty} \frac{A_{1,h}(a,b)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} + O\Bigg(\sum_{\substack{h > x \\ h \in B_k}} h^{-1/k} \Bigg).$$

Again by Remark 2.2 (b) and Lemma 2.5 (c) we have

$$\sum_{\substack{h > x \\ h \in B_k}} h^{-1/k} = O\Bigg(\sum_{\substack{h > x \\ h \in L_{k+2}}} h^{-1/k} \Bigg) = O\Bigg(\sum_{h > x} \frac{l_{k+2}(h)}{h^{1/k}} \Bigg) = O(x^{-2/k(k+2)}).$$

Hence Lemma 2.10 follows.

Lemma 2.11 — For $x \geqslant 1$ and for fixed positive integer n,

$$\sum_{\substack{h \leqslant x \\ (h,n)=1 \\ h \in B_k}} \frac{B_{1,h}(a,b)\, h^{(2k+1)/(k+1)}}{\varphi_2(h)} = \sum_{\substack{h=1 \\ (h,n)=1 \\ h \in B_k}}^{\infty} \frac{B_{1,h}(a,b)\, h^{(2k+1)/k}}{\varphi_2(h)} + O(x^{-1/(k+1)(k+2)}). \qquad \ldots(2.31)$$

PROOF : The proof of Lemma 2.11 is similar to that given in Lemma 2.10.

Remark 2.7 : We write

$$C_k = \zeta^{-1}\left(\frac{2k+2}{k}\right) \sum_{\substack{h=1 \\ (h,n)=1 \\ h\in B_k}}^{\infty} \frac{A_{1,h}(k, k+1)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} \qquad \ldots(2.32)$$

$$C'_k = \zeta^{-1}(2) \sum_{\substack{h=1 \\ (h,n)=1 \\ h\in B_k}}^{\infty} \frac{B_{1,h}(k, k+1)\, h^{(2k+1)/(k+1)}}{\varphi_2(h)} \qquad \ldots(2.33)$$

where $A_{1,h}(a, b)$ and $B_{1,h}(a, b)$ are respectively given by (2.22) and (2.23) we note by Remark 2.2(a), (2.22) and (2.23) that

$$C_2 = \zeta^{-1}(3)\, \zeta(3/2) \quad \text{and} \quad C'_2 = \zeta^{-1}(2)\, \zeta(2/3). \qquad \ldots(2.34)$$

Lemma 2.12 — For $s > 1$

$$\sum_{\substack{m\leqslant x \\ (m,n)=1}} \frac{\mu(m)}{m^s} = \frac{n^s}{\zeta(s)\, \varphi_s(n)} + O\left(\frac{1}{x^{s-1}}\right) \qquad \ldots(2.35)$$

PROOF : We have (cf. Cohen 1961, Lemma 2.3) that

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{m^s} = \frac{n^s}{\zeta(s)\, \varphi_s(n)}$$

Now Lemma 2.12 follows since $\displaystyle\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu(m)}{m^s} = O\left(\sum_{m>x} \frac{1}{m^s}\right) = O\left(\frac{1}{x^{s-1}}\right)$.

Lemma 2.13 — $$M_k(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} q_k(d)\, l_k(\delta)\, \delta. \qquad \ldots(2.36)$$

PROOF : Since $l_k(n)$ and $q_k(n)$ are multiplicative, it follows that the right-hand side sum of (2.36) is a multiplicative function of n (cf. Bateman 1954, Lemma 6.1). $M_k(n)$ is also multiplicative. It therefore suffices to verify (2.36) for $n = p^\alpha$, a prime power and the verification is easy.

3. MAIN RESULTS

In this section we first prove the following :

Theorem 3.1 — If $x \geqslant 2$, then for any fixed positive integer n and for

$$0 < \alpha < \min\left(\frac{2k+1}{4k+4}, \frac{k-1}{k+2}\right),$$

we have

$$L_k(x, n) \equiv \sum_{\substack{m \leqslant x \\ (m,n)=1}} l_k(m) = \frac{x^{1/k} C_k n^{1/k} \varphi(n)\, \varphi_{(k+1)/k}(n)}{\varphi_{(2k+2)/k}(n)} + \frac{x^{1/(k+1)} C'_k n^{1/(k+1)} \varphi(n)\, \varphi_{k/(k+1)}(n)}{\varphi_2(n)} + \Delta_k(x) \quad \ldots(3.1)$$

uniformly, where C_k, C'_k are given by (2.32), (2.33) respectively and

$$\Delta_k(x, n) = \begin{cases} O(\sigma^*(-\alpha, n)\, x^{(\alpha+1)/5}) + O(\sigma^{*2}(-\alpha, n)\, x^{1/6}) & \text{for } k = 2 \\ O(\sigma^{*2}(-\alpha, n)\, x^{1/(k+2)}) & \text{for } k > 2. \end{cases} \quad \ldots(3.2)$$

PROOF: We have by Lemma 2.6,

$$L_k(x, n) \equiv \sum_{\substack{m \leqslant x \\ (m, n)=1}} l_k(m) = \sum_{\substack{m \leqslant x \\ (m, n)=1}} \sum_{\substack{d^k e^{2k+2} f^{k+1} h = m \\ (ef, h)=1 \\ h \in B_k}} \mu(e)$$

From now onwards the condition $h \in B_k$ holds throughout the discussion in the summation formulae and is indicated by writing Σ'. Now

$$L_k(x, n) = \sum'_{\substack{d^k e^{2k+2} f^{k+1} h \leqslant x \\ (d, n)=(e, n)=(f, n)=(h, n) \\ =(e, h)=(f, h)=1}} \mu(e) = \sum_{\substack{d^k e^{2k+2} f^{k+1} h \leqslant x \\ (d, n)=(e, hn)=(f, hn) \\ =(h, n)=1}} \mu(e)$$

$$= \sum'_{\substack{h \leqslant x \\ (h, n)=1}} \sum_{\substack{d^k e^{2k+2} f^{k+1} \leqslant (x/h) \\ (d, n)=(e, hn) \\ =(f, hn)=1}} \mu(e)$$

$$= \sum'_{\substack{h \leqslant x \\ (h, n)=1}} \sum_{\substack{e \leqslant (x/h)^{1/(2k+2)} \\ (e, hn)=1}} \mu(e) \sum_{\substack{d^k f^{k+1} \leqslant x/(he^{2k+2}) \\ (d, n)=(f, hn)=1}} 1$$

$$= \sum'_{\substack{h \leqslant x \\ (h, n)=1}} \sum_{\substack{e \leqslant (x/h)^{1/(2k+2)} \\ (e, hn)=1}} \mu(e)\, T^{n, hn}_{k, k+1}\left(\frac{x}{he^{2k+2}}\right)$$

(*equation continued on p.* 184)

$$= \sum_{\substack{h \leqslant x \\ (h, n)=1}} s_h, \text{ say.} \qquad \ldots(3.3)$$

Now by Lemmas 2.7 and 2.12, we get

$$S_h = \sum_{\substack{e \leqslant (x/h)^{1/(2k+2)} \\ (e, hn)=1}} \mu(e) \left\{ A_{n,hn}(k, k+1) \left(\frac{x}{he^{2k+2}}\right)^{1/k} \right.$$

$$+ B_{n,hn}(k, k+1) \left(\frac{x}{he^{2k+2}}\right)^{1/(k+1)}$$

$$+ O\left(\sigma^*(-\alpha, hn) \left(\frac{x}{he^{2k+2}}\right)^{(\alpha+1)/(2k+1)}\right)$$

$$\left. + O\left(\sigma^*(-\alpha, n)\, \sigma^*(-\alpha, hn) \left(\frac{x}{he^{2k+2}}\right)^{2\alpha/(2k+1)}\right) \right\}$$

$$= A_{n,hn}(k, k+1) \left(\frac{x}{h}\right)^{1/k} \sum_{\substack{e \leqslant (x/h)^{1/(2k+2)} \\ (e, hn)=1}} \mu(e)\, e^{-2(k+1)/k}$$

$$+ B_{n,hn}(k, k+1) \left(\frac{x}{h}\right)^{1/(k+1)} \sum_{\substack{e \leqslant (x/h)^{1/(2k+2)} \\ (e, hn)=1}} \mu(e)\, e^{-2}$$

$$+ O\left(\left(\frac{x}{h}\right)^{(\alpha+1)/(2k+1)} \sigma^*(-\alpha, hn) \times \sum_{e \leqslant (x/h)^{1/(2k+2)}} e^{-2(k+1)(\alpha+1)/(2k+1)}\right)$$

$$+ O\left\{\sigma^*(-\alpha, n)\, \sigma^*(-\alpha, hn) \left(\frac{x}{h}\right)^{2\alpha/(2k+1)} \times \sum_{e \leqslant (x/h)^{1/(2k+2)}} e^{-2\alpha(2k+2)/(2k+1)}\right\}$$

$$S_h = A_{n,hn}(k, k+1) \left(\frac{x}{h}\right)^{1/k} \left\{ \frac{(hn)^{(2k+2)/k}}{\zeta\left(\frac{2k+2}{k}\right) \varphi_{(2k+2)/k}(hn)} \right.$$

$$\left. + O\left(\left(\left(\frac{h}{x}\right)^{1/(2k+2)}\right)^{(k+2)/k}\right) \right\}$$

$$+ B_{n,hn}(k, k+1) \left(\frac{x}{h}\right)^{1/(k+1)} \left\{ \frac{(hn)^2}{\zeta(2)\, \varphi_2(hn)} + O\left(\frac{h}{x}\right)^{1/(2k+2)} \right\} +$$

(*equation continued on p.* 185)

$$+ O\left(\left(\frac{x}{h}\right)^{(\alpha+1)/(2k+1)} \sigma^*(-\alpha, hn) \sum_{e\leqslant(x/h)^{1/(2k+2)}} e^{-2(k+1)(\alpha+1)/(2k+1)}\right)$$

$$+ O\left(\sigma^*(-\alpha, n)\, \sigma^*(-\alpha, hn) \left(\frac{x}{h}\right)^{2\alpha/(2k+1)} \times \sum_{e\leqslant(x/h)^{1/(2k+2)}} e^{-2\alpha(2k+2)/(2k+1)}\right). \qquad \ldots(3.4)$$

By observing that the sum in the third O-term in (3.4) is $O(1)$ and, since $\alpha < (2k+1)/(4k+4)$, the sum in the fourth O-term in (3.4) is $O((x/h)^{(1/(2k+2))-(2\alpha/(2k+1))})$, we get that

$$S_h = \frac{A_{n,hn}(k, k+1)\, x^{1/k} n^{(2k+2)/k} h^{(2k+1)/k}}{\zeta\left(\frac{2k+2}{k}\right) \varphi_{(2k+2)/k}(hn)} + \frac{B_{n,hn}(k, k+1)\, x^{1/(k+1)} n^2 h^{(2k+1)/(k+1)}}{\zeta(2)\, \varphi_2(hn)} + O\left(\sigma^*(-\alpha, hn) \left(\frac{x}{h}\right)^{(\alpha+1)/(2k+1)}\right) + O\left(\sigma^*(-\alpha, n)\, \sigma^*(-\alpha, hn) \left(\frac{x}{h}\right)^{1/(2k+2)}\right). \qquad \ldots(3.5)$$

Now by (3.3), (3.5) and Remark 2.4 we have

$$L_k(x, n) = \frac{x^{1/k} n^{1/k} \varphi(n)\, \varphi_{(k+1)/k}(n)}{\zeta\left(\frac{2k+2}{k}\right) \varphi_{(2k+2)/k}(n)} \sideset{}{'}\sum_{\substack{h\leqslant x\\(h,n)=1}} \frac{A_{1,h}(k, k+1)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} + \frac{x^{1/(k+1)} n^{1/(k+1)} \varphi(n)\, \varphi_{k/(k+1)}(n)}{\zeta(2)\, \varphi_2(n)} \times \sideset{}{'}\sum_{\substack{h\leqslant x\\(h,n)=1}} \frac{B_{1,h}(k, k+1)\, h^{(2k+1)/(k+1)}}{\varphi_2(h)} + O\left(\sigma^*(-\alpha, n)\, x^{(\alpha+1)/(2k+1)} \sideset{}{'}\sum_{h\leqslant x} \frac{\sigma^*(-\alpha, h)}{h^{(\alpha+1)/(2k+1)}}\right) + O\left(\sigma^{*2}(-\alpha, n)\, x^{1/(2k+2)} \sideset{}{'}\sum_{h\leqslant x} \frac{\sigma^*(-\alpha, h)}{h^{1/(2k+2)}}\right). \qquad \ldots(3.6)$$

In case $k = 2$, Theorem 3.1 follows by Remark 2.2(a) and the fact that $h \in B_2$. Suppose $k > 2$. Now, by Remark 2.2(b) and Lemmas 2.3 and 2.4, since $0 < \alpha < (k-1)/(k+2)$, we have

$$O\left(\sigma^*(-\alpha, n)\, x^{(\alpha+1)/(2k+1)} \sum_{h\leqslant x}{}' \frac{\sigma^*(-\alpha, h)}{h^{(\alpha+1)/(2k+1)}}\right)$$

$$+\, O\left(\sigma^{*2}(-\alpha, n)\, x^{1/(2k+2)} \sum_{h\leqslant x}{}' \frac{\sigma^*(-\alpha, h)}{h^{1/(2k+2)}}\right)$$

$$= O(\sigma^{*2}(-\alpha, n)\, x^{1/(k+2)}). \qquad \ldots(3.7)$$

Now from (3.6), (3.7) and Lemmas 2.10 and 2.11 we have

$$L_k(x, n) = \frac{x^{1/k}n^{1/k}\varphi(n)\, \varphi_{(k+1)/k}(n)}{\zeta\left(\frac{2k+2}{k}\right)\varphi_{(2k+2)/k}(n)}$$

$$\times \left\{ \sum_{\substack{h=1\\(h,n)=1}}^{\infty}{}' \frac{A_{1,h}(k, k+1)\, h^{(2k+1)/k}}{\varphi_{(2k+2)/k}(h)} + O(x^{-2/k(k+1)})\right\}$$

$$+ \frac{x^{1/(k+1)}n^{1/(k+1)}\varphi(n)\, \varphi_{k/(k+1)}(n)}{\zeta(2)\, \varphi_2(n)}$$

$$\times \left\{ \sum_{\substack{h=1\\(h,n)=1}}^{\infty}{}' \frac{B_{1,h}(k, k+1)\, h^{(2k+1)/(k+1)}}{\varphi_2(h)} + O(x^{-1/(k+1)(k+2)})\right\}.$$

$$+\, O(\sigma^{*2}(-\alpha, n)\, x^{1/(k+2)}).$$

Now, Theorem 3.1 follows from Remark 2.1, (2.32) and (2.33).

Remark 3.1 : The fact $\alpha > 0$ is used in estimating $L_k(x, n)$ in case $k > 2$. We may take $\alpha = 0$ in case $k = 2$.

Thus we have the following Corollary in virtue of (2.34):

Corollary 3.1.1 ($k = 2$) — For $x \geqslant 2$ and for any fixed positive integer n, we have

$$\sum_{\substack{m\leqslant x\\(m,n)=1}} l(m) = \frac{x^{1/2}n^{1/2}\varphi(n)\, \varphi_{3/2}(n)\, \zeta(3/2)}{\varphi_3(n)\, \zeta(3)}$$

$$+ \frac{x^{1/3}n^{1/3}\varphi_{2/3}(n)\, \zeta(2/3)}{\varphi_2(n)\, \zeta(2)} + O(\theta(n)\, x^{1/5}) + O(\theta^2(n)\, x^{1/6}), \qquad \ldots(3.8)$$

uniformly.

Corollary 3.1.2 — When $k = 2, n = 1$

$$\sum_{n \leqslant x} l(n) = x^{1/2} \frac{\zeta(3/2)}{\zeta(3)} + x^{1/3} \frac{\zeta(2/3)}{\zeta(2)} + O(x^{1/5}). \qquad \text{...(3.9)}$$

Theorem 3.2 — When $x \geqslant 2$, then for any fixed positive integer n and for

$$0 < \alpha < \min\left(\frac{2k+1}{4k+4}, \frac{k-1}{k+2}\right),$$

we have

$$\sum_{\substack{m \leqslant x \\ (m,n)=1}} l_k(m)\, m = \frac{x^{(k+1)/k} C_k n^{1/k} \varphi(n)\, \varphi_{(k+1)/k}(n)}{(k+1)\, \varphi_{(2k+2)/k}(n)}$$

$$+ \frac{x^{(k+2)/(k+1)} C'_k n^{1/(k+1)} \varphi(n)\, \varphi_{k/(k+1)}(n)}{(k+2)\, \varphi_2(n)} + \triangle'_k(x) \qquad \text{...(3.10)}$$

where C_k, C'_k are the constants given by (2.32), (2.33) respectively and

$$\triangle'_k(x, n) = \begin{cases} O(\sigma^*(-\alpha, n)\, x^{(\alpha+6)/5}) + O(\sigma^{*2}(-\alpha, n)\, x^{7/6}) & \text{for } k = 2, \\ O(\sigma^{*2}(-\alpha, n)\, x^{(k+3)/(k+2)}) & \text{for } k > 2. \end{cases} \qquad \text{...(3.11)}$$

PROOF : This follows from Theorem 3.1 and partial summation.

Theorem 3.3 — For $x \geqslant 2$ and for $0 < \alpha < \min\left(\frac{2k+1}{4k+4}, \frac{k-1}{k+2}\right)$, we have

$$\sum_{n \leqslant x} M_k(n) = x^{(k+1)/k} \frac{\alpha_k . C_k}{(k+1)} + x^{(k+2)/(k+1)} \frac{\beta_k . C'_k}{(k+2)} + \triangle''_k(x) \qquad \text{...(3.12)}$$

where α_k, β_k, C_k, C'_k are constants given by (2.28), (2.29), (2.32), (2.33) respectively and

$$\triangle''_k(x, n) = \begin{cases} O(x^{(\alpha+6)/5}), & \text{if } k = 2 \\ O(x^{(k+3)/(k+2)}), & \text{if } k > 2. \end{cases} \qquad \text{...(3.13)}$$

PROOF : We have by Lemma 2.13, Theorem 3.2 and by (2.26) and (2.27),

$$\sum_{n \leqslant x} M_k(n) = \sum_{n \leqslant x} \sum_{\substack{d\delta = n \\ (d,\delta)=1}} q_k(d)\, l_k(\delta)\, \delta = \sum_{\substack{d\delta \leqslant x \\ (d,\delta)=1}} q_k(d)\, l_k(\delta)\, \delta$$

$$= \sum_{d \leqslant x} q_k(d) \sum_{\substack{\delta \leqslant (x/d) \\ (\delta, d)=1}} l_k(\delta)\, \delta$$

(*equation continued on p.* 188)

$$= \sum_{d \leqslant x} q_k(d) \left\{ \frac{(x/d)^{(k+1)/k} C_k d^{1/k} \varphi(d)\, \varphi_{(k+1)/k}(d)}{(k+1)\, \varphi_{(2k+2)/k}(d)} \right.$$

$$+ \frac{(x/d)^{(k+2)/(k+1)} C'_k d^{1/(k+1)} \varphi(d)\, \varphi_{k/(k+1)}(d)}{(k+2)\, \varphi_2(d)}$$

$$\left. + \Delta'_k \left(\frac{x}{d}, d \right) \right\}$$

$$= \frac{x^{(k+1)/k} C_k}{(k+1)} \sum_{m \leqslant x} \frac{q_k(m)\, \varphi(m)\, \varphi_{(k+1)/k}(m)}{m \varphi_{(2k+2)/k}(m)}$$

$$+ \frac{x^{(k+2)/(k+1)} C'_k}{(k+2)} \sum_{m \leqslant x} \frac{q_k(m)\, \varphi(m)\, \varphi_{k/(k+1)}(m)}{m \varphi_2(m)}$$

$$+ \sum_{m \leqslant x} q_k(m)\, \Delta'_k \left(\frac{x}{m}, m \right)$$

$$= \frac{x^{(k+1)/k} C_k}{(k+1)} \{\alpha_k + O(x^{-1/k})\}$$

$$+ \frac{x^{(k+2)/(k+1)} C'_k}{(k+2)} \{\beta_k + O(x^{-1/(k+1)})\}$$

$$+ \sum_{m \leqslant x} q_k(m)\, \Delta'_k \left(\frac{x}{m}, m \right).$$

Hence

$$\sum_{n \leqslant x} M_k(n) = \frac{x^{(k+1)/k} \alpha_k C_k}{(k+1)} + \frac{x^{(k+2)/(k+1)} \beta_k C'_k}{(k+2)} + O(x)$$

$$+ \sum_{m \leqslant x} q_k(m)\, \Delta'_k \left(\frac{x}{m}, m \right). \qquad \ldots(3.14)$$

It follows from (3.11) that

$$\sum_{m \leqslant x} q_k(m)\, \Delta'_k \left(\frac{x}{m}, m \right) = \begin{cases} O(x^{(\alpha+6)/5}), & \text{if } k = 2, \\ (x^{(k+3)/(k+2)}, & \text{if } k \geqslant 3. \end{cases} \qquad \ldots(3.15)$$

Now, Theorem 3.3 follows from (3.14) and (3.15).

By Theorem 3.3 and Remarks 2.7, 3.1, we have the following :

Corollary 3.3.1 ($k = 2$) — If $M(n)$ denote the maximal square-full divisor of n, then for $x \geqslant 2$

$$\sum_{n \leqslant x} M(n) = \frac{\alpha_2 \zeta(3/2)}{3\zeta(3)} x^{3/2} + \frac{\beta_2 \zeta(2/3)}{4\zeta(2)} x^{4/3} + O(x^{6/5}). \qquad ...(3.16)$$

Remark 3.3 : Every positive integer n is uniquely factorable in the form

$$n = M_k(n) \gamma_k^*(n) \qquad ...(3.17)$$

where $\gamma_k^*(n)$ denote the maximal unitary k-free divisor of n.

Theorem 3.4 — For $x \geqslant 2$ and $0 < \alpha < \min\left(\frac{2k+1}{4k+4}, \frac{k-1}{k+2}\right)$, we have

$$\sum_{n \leqslant x} \frac{1}{\gamma_k^*(n)} = \alpha_k C_k x^{1/k} + \beta_k C_k' x^{1/(k+1)} + \Delta_k'''(x) \qquad ...(3.18)$$

where

$$\Delta_k'''(x) = \begin{cases} O(x^{(\alpha+1)/5}), & \text{if } k = 2, \\ O(x^{1/(k+2)}), & \text{if } k > 2. \end{cases} \qquad ...(3.19)$$

PROOF : This theorem follows from Remark 3.3, Theorem 3.3 and partial summation.

Corollary 3.4.1 ($k = 2$) — If $\gamma^*(n)$ denote the maximal unitary, square-free divisor of n, then

$$\sum_{n \leqslant x} \frac{1}{\gamma^*(n)} = \frac{\alpha_2 \zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\beta_2 \zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}). \qquad ...(3.20)$$

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