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GRAPHS WHOSE SQUARES ARE CHORDAL

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A simple graph G is called chordal if every cycle C_k in G of length $k > 3$ has a chord. A sufficient condition is obtained in order that the square of a graph is chordal.

For terminology and notation in graph theory, we refer the reader to the book by Harary (1971).

An undirected simple graph G is called chordal if every cycle C_k of length $k > 3$ in G has a chord i.e., an edge of G joining two non-adjacent points of C_k. The following question was posed by Laskar (1980): "For which graphs G, the squares G^2 are chordal?". In an attempt to answer this question we give in this note a sufficient condition for the square of a graph to be chordal.

The following two properties of chordal graphs are trivially true:

(i) Every induced subgraph of a chordal graph is chordal, and

(ii) Each cycle of a chordal graph is 'triangulated' i.e., the interior face of any cycle is divided into triangles.

We observe that even if G is chordal, G^2 need not be chordal and that the square of a non-chordal graph may as well be chordal.

Theorem — If G has no induced subgraph isomorphic to $K_{1,3}$ or cycle C_n $(n \geqslant 6)$ then G^2 is chordal.

Proof : Suppose the theorem is not true. Then there exists a chordless cycle $C_m = (v_1 v_2 \ldots v_m v_1)$, where $m \geqslant 4$, in G^2. We claim that at least one edge of C_m must be an edge of G. If this is false, each edge $v_i v_{i+1}$ of C_m must arise out of two edges $v_i u_i$ and $u_i v_{i+1} (v_{n+1} = v_1)$ of G. Further, the points u_i are all distinct. (If, for instance, $u_i = u_j$ then $v_i v_j$ will be a chord of C_m in G^2, a contradiction). But then we get a cycle $Z = (v_1 u_1 v_2 \ldots v_m u_m v_1)$ of length $2m \geqslant 8$ in G.

As by hypothesis G has no induced cycle of length $\geqslant 6$, Z cannot be an induced cycle and hence Z must have a chord. If this chord joins u_i and v_j then there would result a chord of C_m in G^2, a contradiction to our assumption. On the other hand

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if this chord joins u_i and u_j, then a $K_{1,3}$ would be induced at u_i (also, at u_j), a contradiction to the hypothesis. Thus C_m must contain at least one edge of G.

Suppose C_m has $r(\geqslant 1)$ edges of G. Clearly, no two of these edges can be consecutive in C_m and so $r \leqslant [m/2]$, where $[x]$ denotes the integral part of x. Replacing each edge of G^2 in C_m not in G by a path of length 2 in G, there results a cycle of length $2(m - r) + r = 2m - r$. As $r \leqslant [m/2]$ and $m \geqslant 4$, $2m - r \geqslant 6$. Arguing as before we conclude that this cycle must contain a chord hence inducing a $K_{1,3}$ in G, again contradicting our hypothesis. This proves the theorem.

Remark : The conclusion given in the above theorem is only sufficient but not necessary. For example, the wheel $W_n = K_1 + C_{n-1}(n \geqslant 7)$ has both $K_{1,3}$ and $C_n(n \geqslant 6)$ as induced subgraphs though W_n^2 is chordal.

Corollary — If G is chordal and free from $K_{1,3}$ as an induced subgraph then G^2 is chordal.

References

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