Indian J. pure appl. Math., 12(2) : 195–199, February 1981

ON OUTERPLANAR REPEATED LINE GRAPHS

V. R. KULLI

Department of Mathematics, Karnatak University Post Graduate Centre, Gulbarga

AND

D. G. AKKA

Department of Mathematics, S. S. Margol College, Shahabad, Gulbarga

(*Received* 24 *July* 1980)

Chartrand *et al.* (1971) have established a characterization of graphs whose line graphs are outerplanar. The main results of this paper are characterizations of graphs whose repeated line graphs are outerplanar. In addition, we give characterizations of graphs with outerplanar line graphs in terms of forbidden subgraphs.

1. INTRODUCTION

Definitions not given here may be found in Behzad and Chartrand (1972). The graphs under consideration are ordinary graphs, *i.e.*, finite, undirected graphs possessing no loops or multiple edges. Chartrand *et al.* (1971) have obtained the following and it is useful in the proof of our main result:

Theorem A — The line graph $L(G)$ of a graph G is outerplanar if and only if $\Delta(G) \leqslant 3$ and if $\deg v = 3$ for a vertex v of G, then v is a cutvertex.

2. OUTERPLANAR GRAPHS

Theorem 1 — The second line graph $L^2(G)$ of a planar graph G is outerplanar if and only if G satisfies the following conditions:

(i) $\deg v \leqslant 3$ for every vertex v of G,

(ii) $\deg u + \deg v \leqslant 5$ for every edge (u, v) of G and

(iii) if $\deg u + \deg v = 5$, then (u, v) is a bridge of G.

PROOF : Suppose $L^2(G)$ is outerplanar. Then $L(G)$ is necessarily outerplanar. Theorem A implies that $\deg v \leqslant 3$ for every vertex v of G. This proves (i). To prove (ii), assume $\deg u + \deg v \geqslant 6$ for some edge (u, v) of G. Then in $L(G)$ the vertex uv has degree $\deg u + \deg v - 2 \geqslant 4$, which is a contradiction to Theorem A. To prove (iii), suppose $\deg u + \deg v = 5$ for some edge (u, v) of G. Then in $L(G)$, $\deg uv = \deg u + \deg v - 2 = 3$. By Theorem A, uv is a cutvertex of $L(G)$. Hence (u, v) is a bridge of G.

To prove the converse, assume (i), (ii) and (iii). We prove first that $L(G)$ is outerplanar. By (i), $\deg v \leqslant 3$ for every vertex v of G. By (ii), if $\deg v = 3$, then $\deg u \leqslant 2$ for every vertex u adjacent to v. If $\deg u = 2$ for some such u of G, then by (iii), v is a cutvertex of G. If $\deg u = 1$ for every such u, then again v is a cutvertex. Theorem A implies that $L(G)$ is outerplanar.

We now prove that $L^2(G)$ is outerplanar. By (ii), the degree of every vertex uv in $L(G)$ is $\deg u + \deg v - 2 \leqslant 3$. If $\deg uv = 3$ for some vertex uv of $L(G)$, then $\deg u + \deg v = 5$ in G and by (iii), uv is a cutvertex of $L(G)$. By Theorem A, $L^2(G)$ is outerplanar.

Theorem 2 — The *n*th line graph $L^n(G)$ $(n \geqslant 3)$ of a planar graph G is outerplanar if and only if G satisfies the following conditions:

(i) $\deg v \leqslant 3$ for every vertex v of G and

(ii) if $\deg v = 3$ for some vertex v of G, then the component of G containing v is $K_{1,3}$.

PROOF : We first prove this theorem for $n = 3$. Suppose $L^3(G)$ is outerplanar. Then clearly $L^2(G)$ is outerplanar. By Theorem 1, $\deg v \leqslant 3$ for every vertex v of G. This proves (i). Let $\deg v = 3$ for some vertex v of G and Let $e_i = (v, v_i)$, $i = 1, 2, 3$, be the edges adjacent to v. If the component of G containing v is not $K_{1,3}$, then there exists at least one edge incident to v_i, say v_1. Let this edge be $e_4 = (v_1, v_4)$. Clearly (e_1, e_4) becomes an edge in $L(G)$ which is incident to the triangle formed by the vertices e_1, e_2 and e_3. Thus in $L(G)$, $\deg e_1 + \deg e_2 = 5$ and $\deg e_1 + \deg e_3 = 5$. But (e_1, e_2) and (e_1, e_3) are not bridges of $L(G)$. Theorem 1 implies that $L^3(G)$ is not outerplanar, which is a contradiction.

For the converse, assume (i) and (ii). By (i), $\deg v \leqslant 3$ for every vertex v of G. By (ii), if $\deg v = 3$ for some vertex v of G, then the component of G containing v is $K_{1,3}$. Clearly $L^n(G)$, $n \geqslant 1$, are all outerplanar graphs.

For the general case, assume $L^n(G)$ $(n \geqslant 3)$ is outerplanar. Then in particular, $L^3(G)$ is planar and hence (i) and (ii) are true as above. The converse part of the general case has already been proved above.

3. Forbidden Subgraphs

Theorem 3 — A graph G has an outerplanar line graph if and only if it has no subgraph homeomorphic to $K_{1,4}$ or $K_4 - x$.

PROOF : Let G be a graph with an outerplanar line graph. If H is a subgraph of G, then H has an outerplanar line graph. All that we need to do is to show that graphs homeomorphic to $K_{1,4}$ or $K_4 - x$ have nonouterplanar line graphs. This follows from Theorem A since graphs homeomorphic to $K_{1,4}$ have $\Delta(G) > 3$, and graphs homeomorphic to $K_4 - x$ have vertices of degree 3 which are not cutvertices.

Conversely, suppose a graph G has no subgraph homeomorphic to $K_{1,4}$ or $K_4 - x$. If $\Delta(G) > 3$, then G has a subgraph homeomorphic to $K_{1,4}$, which is a contradiction.

Suppose G has a noncutvertex u of degree 3. Let u_1, u_2 and u_3 be the vertices adjacent to u. Since u is not a cutvertex, there exists a path P_1 from u_1 to u_2 that does not contain u. Two different possible cases arise.

Case 1 — u_3 is on P_1. Then G has a subgraph homeomorphic to $K_4 - x$ [see Fig. 1(a)].

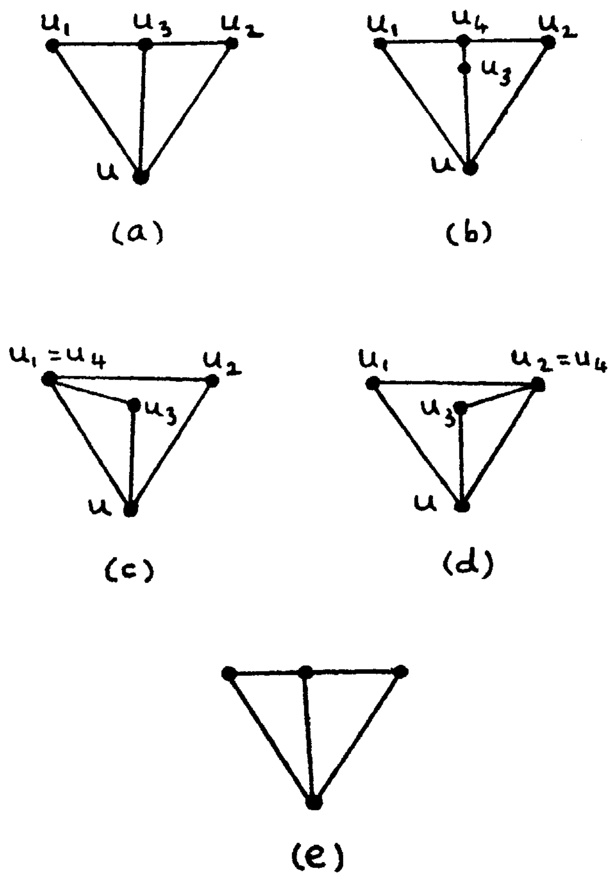

FIG. 1.

Case 2 — u_3 is not on P_1. Since u is not a cutvertex, there is a path P_2 from u_3 to u_1 or u_2 not containing u. Without loss of generality we assume P_2 is a path from u_3 to u_1. Let u_4 be the last vertex of P_1 (when moving from u_1) which also belongs to P_2. If $u_4 = u_1$ or u_2 or u_3 then G contains a subgraph homeomorphic to $K_4 - x$ [see Fig. 1(c) or (d) or (e)]. If $u_4 \neq u_1$, u_2 and u_3, then G has a subgraph homeomorphic to $K_4 - x$ [see Fig. 1(b)].

In both cases G has a subgraph homeomorphic to $K_4 - x$. We come to conclusion that if u is a vertex of degree three in G, then u is a cutvertex. Thus, Theorem A implies that G has an outerplanar line graph.

Before presenting the next Theorem we consider the graph $K_{3,3} - C_4$ which is a spanning subgraph of $K_{3,3}$ obtained by removal of edges of C_4.

Theorem 4 — A graph G has an outerplanar second line graph $L^2(G)$ if and only if it has no subgraph homeomorphic to $K_{1,4}$, $K_{3,3} - C_4$ or $K_3.K_2$ (see Fig. 2).

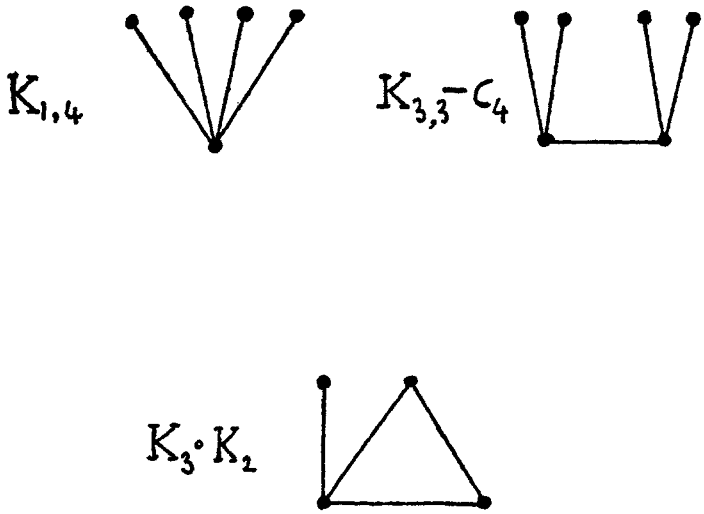

FIG. 2.

PROOF : The necessity follows from Theorem 1, since $\Delta(K_{1,4}) = 4$, $K_{3,3} - C_4$ contains two adjacent vertices such that the sum of the degrees of the vertices is 6 and $K_3.K_2$ contains an edge (u, v) such that $\deg u + \deg v = 5$, which is not a bridge of G.

For the sufficiency, assume G contains no subgraph homeomorphic to $K_{1,4}$, $K_{3,3} - C_4$ or $K_3.K_2$. The $\Delta(G) \leqslant 3$ since otherwise G would contain $K_{1,4}$ as a subgraph. Assume $\deg u + \deg v \geqslant 6$ for some edge (u, v) of G. Since $\Delta(G) \leqslant 3$ each of u and v has degree 3. Let (u, u_i) and (v, v_j), $i, j = 1, 2$ be the edges of G. If $u_i \neq v_j$ for $i, j = 1, 2$, then $K_{3,3} - C_4$ is a subgraph of G. If $u_i = v_j$ for some i, j then $K_3.K_2$ is a subgraph of G. Therefore condition (ii) of Theorem 1 holds for G. Assume there exists an edge (u, v) of G, such that $\deg u + \deg v = 5$ and

(u, v) not a bridge of G. Then $\deg u = 3$ and $\deg v = 2$ (or $\deg u = 2$ and $\deg v = 3$). But then $K_3.K_2$ is a subgraph of G. Hence condition (iii) of Theorem 1 holds for G. By Theorem 1, $L^2(G)$ is outerplanar.

Theorem 5 — A graph G has an outerplanar nth line graph $L^n(G)$ $(n \geqslant 3)$ if and only if it has no subgraph homeomorphic to $K_{1,4}$, $K_{1,3}.K_2$ or $K_3.K_2$ (see Fig. 3).

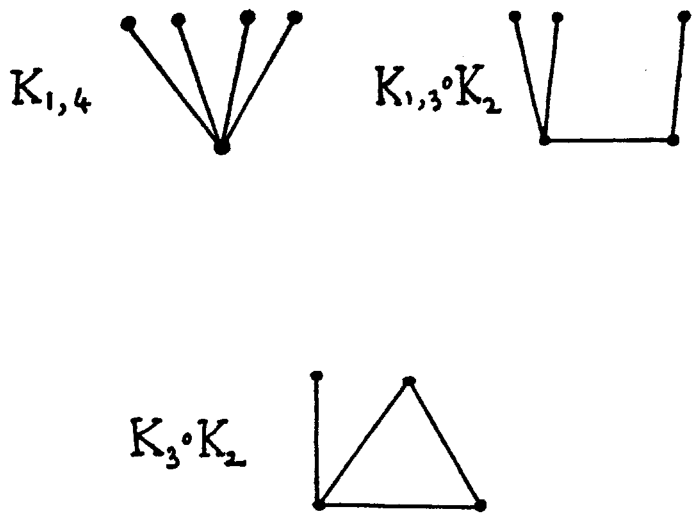

FIG. 3.

PROOF: Since $\triangle(K_{1,4}) = 4$ and each of $K_{1,3}.\ K_2$ and $K_3.K_2$ contains a vertex of degree 3 and each is not $K_{1,3}$. It follows from Theorem 2 that if $L^n(G)$ $(n \geqslant 3)$ is outerplanar, then it cannot contain a homeomorph to any of these forbidden subgraphs.

Conversely, suppose G is a graph which does not contain a subgraph homeomorphic to either $K_{1,4}$, $K_{1,3}.K_2$ or $K_3.K_2$. Then $\triangle(G) \leqslant 3$. Now let $\deg v = 3$ for some vertex v of G. Let the three edges adjacent to v be (v, v_i), $i = 1, 2$, and 3. If the component of G containing v is not $K_{1,3}$, then there exists either an edge which is adjacent to one of v_i or a path between any two of v_i. This implies that G has a subgraph homeomorphic to $K_{1,3}.K_2$ or $K_3.K_2$. The contradiction proves that a component of G containing v is $K_{1,3}$. Theorem 2 implies that $L^n(G)$ $(n \geqslant 3)$ is outerplanar.

REFERENCES

Behzad, M., and Chartrand, G. (1972). Introduction to the Theory of Graphs. Allyn and Bacon, Inc., Boston.

Chartrand, G., Geller, D., and Hedetniemi, S. (1971). Graphs with forbidden subgraphs. *J. Combinatorial Theory*, **10**, 12–41.