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CONDITIONS IMPLYING UNITICITY

R. P. SINGH AND N. SINGH

Department of Mathematics, Kurukshetra University, Kurukshetra (Haryana)

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The main object of this paper is to prove the following results:

Theorem 1 — If T is an invertible operator in $C_\rho (\rho \geqslant 1)$ with T^n unitary for some integer $n \geqslant 1$, then T is unitary.

Theorem 2 — Let $T \in B(H)$ be an invertible operator, and suppose that there exists an operator $S \in B(H)$ such that $0 \notin \overline{W(S)}$ and $T^* = S^{-1} T^{-n} S$ for an integer $n \geqslant 1$. If T and T^{-1} are ρ-oid ($\rho \geqslant 1$) and δ-oid ($\delta \geqslant 1$) respectively, then T is unitary.

1. INTRODUCTION

In this paper, an operator means a bounded linear operator on a complex Hilbert space. For a Hilbert space H, let $B(H)$ be the Banach algebra of all operators on H. For an operator $T \in B(H)$, the symbols $\sigma(T)$, co $\sigma(T)$, $W(T)$, $\overline{W(T)}$, $r(T)$ and $w(T)$ stand for the spectrum of T, the convex hull of $\sigma(T)$, the numerical range of T, the closure of $W(T)$, the spectral radius of T and the numerical radius of T respectively.

An operator $T \in B(H)$ is said to be normaloid if $\|T\| = r(T)$, spectraloid if $w(T) = r(T)$ and convexoid if $\overline{W(T)} = \text{co}\, \sigma(T)$. Let C_ρ $(\rho > 0)$ be the class of all operators with unitary ρ-dilation in the sense of Sz-Nagy and Foias (1970). Holbrook (1968) has defined the operator radius $w_\rho(T)$ $(0 < \rho < \infty)$ of T as follows:

$$w_\rho(T) = \inf \{u : u > 0, u^{-1} T \in C_\rho\}.$$

Moreover he has proved that $T \in C_\rho$ iff $w_\rho(T) \leqslant 1$. An operator $T \in B(H)$ is said to be ρ-oid if $w_\rho(T^n) = (w_\rho(T))^n$, $(n = 1, 2, 3, \ldots)$ and for each $\rho \geqslant 1$, $w_\rho(T) = r(T)$ iff T is ρ-oid (Furuta 1969). Clearly 1-oid and 2-oid operators are normaloid and spectraloid respectively. $T \in B(H)$ is called k-paranormal $(k \geqslant 2)$ if $\|Tx\|^k \leqslant \|T^k x\| \, \|x\|^{k-1}$ for all $x \in H$ and 2-paranormal operator is called paranormal. Recently, Juneja (1977) has defined a new class $(N^* : k)$ of operators as follows:

$$T \in (N^* : k) \text{ if } \|T^* x\|^k \leqslant \|T^k x\| \, \|x\|^{k-1} \text{ for all } x \in H.$$

It is known (Istratescu and Istratescu 1967, Juneja 1977) that the class of k-paranormal operators and the class $(N^* : k)$ are contained in the class of normaloid operators.

Kato and Moriya (1977) introduced the Campbell equivalence of operators S and $T \in B(H)$, denoted by $S \sim T$ as follows:

$$S \sim T \text{ iff } S^* S = T^* T \text{ and } S^* + S = T^* + T.$$

2. Operators of Class C_ρ ($\rho \geqslant 1$) and Campbell Equivalence

Our purpose in this section is to generalize the following theorem and its corollary (Nakamoto 1977) for the operators of class $C_\rho(\rho \geqslant 1)$ and ρ-oid ($\rho \geqslant 1$) operators respectively.

Theorem A — If T is a contraction with $T \sim T^{-1}$, then T is unitary.

Corollary — If T is normaloid with $T \sim T^{-1}$, then T is unitary.

First we establish a more general theorem.

Theorem 1 — If T is an invertible operator of class $C_\rho(\rho \geqslant 1)$ with T^n unitary for some integer $n \geqslant 1$, then T is unitary.

Before we prove our Theorem we quote the following well-known results as lemmas:

Lemma 1 (Holbrook 1968, Theorem 6.5) — Suppose $T, S \in B(H)$ are commuting operators and that T is normal. Then, for all ($\rho \geqslant 1$),

$$w_\rho(T S) \leqslant w_\rho(T)\, w_\rho(S).$$

Lemma 2 (Stampfli 1969, Corollary 4) — If $T \in C_\rho(\rho \geqslant 1)$, $T^{-1} \in C_\delta$ ($\delta \geqslant 1$), then T is unitary.

Proof of Theorem 1 : By hypothesis, T^n is unitary for some integer $n \geqslant 1$. Therefore,

$$\begin{aligned} w_\rho(T^{-1}) &= w_\rho(T^{-n}\, T^{n-1}) \leqslant w_\rho(T^{-n})\, w_\rho(T^{n-1}), \text{ by Lemma 1} \\ &= \|T^{-n}\|\, w_\rho(T^{n-1}) = w_\rho(T^{n-1}) \end{aligned}$$

$\leqslant w_\rho(T)^{n-1} \leqslant 1$, by power inequality of Holbrook. Therefore, T is unitary by Lemma 2.

We will now prove the following corollaries which include Theorem A and its corollary as special cases.

Corollary 1 — If T is an invertible operator in $C_\rho(\rho \geqslant 1)$ with $T^n \sim T^{-n}$ for some integer $n \geqslant 1$, then T is unitary.

Proof : $T^n \sim T^{-n}$ implies $T^{*n}\, T^n = T^{*-n}\, T^{-n}$. Therefore, $T^{*2n} = T^{-2n}$. So T^{2n} is unitary and hence T is unitary by Theorem 1.

Corollary 2 — If T is an invertible ρ-oid ($\rho \geqslant 1$) operator with $T^n \sim T^{-n}$ for some integer $n \geqslant 1$, then T is unitary.

PROOF : As in the above corollary, T^{2n} is unitary. Therefore, the spectrum of T lies on the unit circle. We have $w_\rho(T) = r(T) = 1$, since T is ρ-oid by hypothesis. Therefore, $T \in C_\rho$ and hence T is unitary by Corollary 1.

Nakamoto has also given the following characterization of unitary operators:

Theorem B — An operator T is unitary if and only if $T^* \sim T^{-1}$. However, $T^* \sim T^{-1}$ cannot be replaced by $T^{*n} \sim T^{-n}$ for some integer $n \geqslant 2$. We give an example of a non-unitary operator T on the 2-dimensional space which satisfies $T^{*2} \sim T^{-2}$.

Example — Let

$$T = \begin{pmatrix} 0 & 3i \\ 1/3i & 0 \end{pmatrix}$$

It is easily seen that $T^{*2} \sim T^{-2}$, but T is non-unitary. In the following corollary we impose a restriction on T by which $T^{*n} \sim T^{-n}$ for some integer $n \geqslant 2$ implies that T is unitary.

Corollary 3 — If T is an invertible operator in $C_\rho(\rho \geqslant 1)$ with $T^{*n} \sim T^{-n}$ for some integer $n \geqslant 2$, then T is unitary.

PROOF : $T^{*n} \sim T^{-n}$ implies $(T^{*n})^* T^{*n} = T^{*-n} T^{-n}$ or $(T^n T^{*n})^2 = 1$. This implies that $T^n T^{*n} = 1$ and hence T^n is unitary. Again by Theorem 1, T is unitary.

Remark 1 : Putting $\rho = n = 1$ in Corollaries 1 and 2 we have Theorem *A* and its corollary respectively.

3. SIMILARITY OF OPERATORS

In this section we shall give a theorem which includes the following result due to Deprima (1974).

Theorem C — Let $T, S \in B(H)$ with T invertible. If

(i) $ST^* = T^{-1}S$ with $0 \notin \overline{W(S)}$,

(ii) T is either convexoid or normaloid,

(iii) T^{-1} is either convexoid or normaloid, then T is unitary.

Theorem 2 — Let $T \in B(H)$ be an invertible operator, and suppose that there exists an invertible operator $S \in B(H)$ such that $0 \notin \overline{W(S)}$ and $T^* = S^{-1} T^{-n} S$ for an integer $n \geqslant 1$. If T and T^{-1} are ρ-oid ($\rho \geqslant 1$) and δ-oid ($\delta \geqslant 1$), respectively, then T is unitary.

For the proof of Theorem 2, the following lemma is required:

Lemma 3 (Patel 1973, Theorem 5) — Let T be a left invertible operator with a left inverse T_1. If there exists an operator S such that $T^* = S^{-1} T_1^n S$, $0 \notin \overline{W(S)}$ and n is a positive integer, then $\sigma(T)$ lies in the unit disk.

Proof of Theorem 2 — By Lemma 3, $\sigma(T)$ lies in the unit disc. Therefore, $r(T) \leqslant 1$ and $w_\rho(T) = r(T) \leqslant 1$, T being ρ-oid. Again since $r(T^*) = r(T)$, $\sigma(T^*)$ lies in the unit disc. By hypothesis T^{-n} is similar to T^* and similar operators have the same spectrum, so $\sigma(T^{-1})^n = \sigma(T^{-n})$ implies that $\sigma(T^{-1})$ lies in the unit disc. Hence $w_\delta(T^{-1}) = r(T^{-1}) \leqslant 1$ since T^{-1} is δ-oid by hypothesis. Therefore, T is unitary by Lemma 2.

The following (simple) lemma is required in the proof of the corollary to follow.

Lemma 4 — If T is a k-paranormal operator or in the class $(N^* : k)$ $(k \geqslant 2)$ and has a right inverse, then T is invertible.

Corollary 4 — Let $T \in B(H)$ be a left invertible operator with a left inverse T_1 and, suppose that there exists an operator $S \in B(H)$ such that $0 \notin \overline{W(S)}$ and $T^* = S^{-1} T_1^n S$ for a positive integer n. Then each of the following conditions implies that T is unitary:

(i) T_1 or T^* is paranormal,

(ii) T is ρ-oid ($\rho \geqslant 1$) and T_1 is k-paranormal,

(iii) T_1 is ρ-oid ($\rho \geqslant 1$) and T^* is k-paranormal,

(iv) T is ρ-oid ($\rho \geqslant 1$) and $T_1 \in (N^* : k)$,

(v) T_1 is ρ-oid ($\rho \geqslant 1$) and $T^* \in (N^* : k)$

Remark 2 : (i) differs from (ii) - (v) due to the fact that an invertible paranormal has a paranormal inverse.

Proof : The proof follows readily from Lemma 4 and Theorem 2.

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