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ON A GENERALIZATION OF IYER'S SPACE OF INTEGRAL FUNCTIONS III

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The author extends the results of Ganapathy Iyer concerning metrically bounded functionals and metrically bounded transformations for the space of entire functions over the field of complex numbers to a more general class of spaces.

1. INTRODUCTION

The space of integral functions over the field of complex numbers was introduced by Ganapathy Iyer (1948). For any set X and a function $s : X \to N$, N being the set of non-negative integers, we define

$$A(X, C, s) = \{\alpha : X \to C \mid \text{for every } \epsilon > 0 \text{ there exists only finitely many } x \in X \text{ such that } |\alpha(x)|^{t(x)} \geqslant \epsilon\}, \qquad \ldots(1.1)$$

t being defined as $t(x) = 1/s(x)$ if $s(x) \neq 0$ and $= 1$ otherwise. It was shown by the author (Kankurikar 1978) that $A(X, C, s)$ is a complete metric linear space with respect to the metric arising from $\| \cdot \|$ defined on $A(X, C, s)$ as

$$\| \alpha \| = \sup_{x \in X} |\alpha(x)|^{t(x)} \qquad \ldots(1.2)$$

$\| \cdot \|$ satisfying the following properties:

$$\| \alpha \| \geqslant 0 \text{ and } \| \alpha \| = 0 \text{ if and only if } \alpha = 0 \qquad \ldots(1.3)$$

$$\| \alpha + \beta \| \leqslant \| \alpha \| + \| \beta \| \qquad \ldots(1.4)$$

$$\| \lambda\alpha \| \leqslant A(\lambda) \| \alpha \| \qquad \ldots(1.5)$$

where $\lambda \in C$ and $A(\lambda) = \max \{1, |\lambda|\}$.

It is easily proved that every element α of $A(X, C, s)$ can be expressed as

$$\alpha = \sum_{i=1}^{\infty} \alpha(x_i) . x_i \qquad \ldots(1.6)$$

for some sequence (x_n) (depending on α) in X, each $x \in X$ being identified with the element of $A(X, C, s)$ defined by $x(y) = 1$ if $y = x$ and $x(y) = 0$ otherwise.

In this paper we extend the results of Ganapathy Iyer (1956) concerning metrically bounded functionals and metrically bounded transformations.

2. Metrically Bounded Functionals

Definition 2.1 (Ganapathy Iyer 1956) — A linear functional $f : \Gamma \to C$ will be called a metrically bounded functional if there exists a positive real number K such that

$$| f(\alpha) | \leqslant K | \alpha | \text{ for every } \alpha \in \Gamma.$$

We denote by Γ^* the set of all metrically bounded linear functionals on Γ. Γ^* is a vector space with respect to the natural operations of addition and scalar multiplication. We can also define $\| . \|$ on Γ^* as

$$\| f \| = \inf \{K \mid | f(\alpha) | \leqslant K | \alpha | \text{ for all } \alpha \in \Gamma\} \quad ...(2.1)$$

$\{\delta_n \mid n \geqslant 0\}$ is a proper base for Γ (see Ganapathy Iyer 1956). The following theorem is due to Sisarcick (1975).

Theorem 2.1 — A linear functional $F : \Gamma \to C$ is metrically bounded if and only if there are complex numbers a and b such that if $f = \sum_{n=0}^{\infty} a_n \delta_n \in \Gamma$ then $F(f) = a_0 a + a_1 b$ and in that case $\| F \| = | a | + | b |$. In fact $a = F(\delta_0)$, $b = F(\delta_1)$.

The following corollary follows immediately from the Theorem 2.1.

Corollary 2.1 — Γ^* is isometric isomorphic to the Banach space $C \times C$ with $\| . \|$ defined on it as

$$\| (a, b) \| = | a | + | b | .$$

Proof : It is easily verified that the map

$$\zeta : \Gamma^* \to C \times C$$

$$f \to (f(\delta_0), f(\delta_1)).$$

is an isometric isomorphism.

Definition 2.2 — For any set Y by $L(Y)$ we mean the following Banach space

$$L(Y) = \{\beta : Y \to C \mid \text{ there exists a number } K \text{ having the property that } \sum_{n=1}^{\infty} | \beta(x_n) | \leqslant K \text{ for any } (x_n) \text{ in } Y\} \quad ...(2.2)$$

the norm on $L(Y)$ being defined as

$$\| \beta \| = \sup \left\{ \sum_{n=1}^{\infty} | \beta(x_n) | \,\Big|\, (x_n) \subset Y \right\}. \qquad \ldots(2.3)$$

We now study the space $A(X, C, s)^*$ of metrically bounded linear functionals on $A(X, C, s)$. The following theorem gives the structure of $A(X, C, s)^*$.

Theorem 2.2 — $A(X, C, s)^* \cong L(Y)$, where $Y = s^{-1}(0) \cup s^{-1}(1)$. In other words, to each $\beta \in L(Y)$ there exists a unique $f \in A(X, C, s)^*$ such that $\| f \| = \| \beta \|$ and conversely.

PROOF : Let $\beta : Y \to C$ be an element of $L(Y)$. We define f on $A(X, C, s)$ as follows:

$$f(x) = \beta(x) \text{ if } x \in Y$$
$$= 0 \quad \text{if } x \in X - Y.$$

We need to prove now that $f \in A(X, C, s)^*$ and $\| f \| = \| \beta \|$. Let

$$\alpha = \sum_{i=1}^{\infty} \alpha(x_i)\, x_i \in A(X, C, s).$$

Then

$$| f(\alpha) | = \left| \sum_{i=1}^{\infty} \alpha(x_i)\, f(x_i) \right|$$

$$= \left| \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} \alpha(x_i)\, f(x_i) \right|$$

$$\leqslant \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} | \alpha(x_i) | \; | \beta(x_i) |$$

$$\leqslant \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} | \beta(x_i) | \,.\, \sup_{x_i \in Y} | \alpha(x_i) |$$

$$\leqslant \| \beta \| \, \| \alpha \|.$$

Hence we have proved that f is metrically bounded linear functional and that $\| f \| \leqslant \| \beta \|$.

To prove the reverse inequality it suffices to prove that for any sequence (x_n) in Y, $\sum_{n=1}^{\infty} | \beta(x_n) | \leqslant \| f \|$.

For any $c \in C$ we define $\dot{c}$ as $\dot{c} = \dfrac{\bar{c}}{| c |}$ if $| c | \neq 0$ and $= 0$ otherwise. Let (x_n) be any sequence in y. For each $p \geqslant 1$ we define

$$\alpha_p = \sum_{i=1}^{p} \beta(\hat{x}_i) . x_i$$

Obviously $\| \alpha_p \| = 1$ for every p. Since f is metrically bounded functional we have

$$| f(\alpha_p) | \leqslant \| f \| \quad \text{for every} \quad p \geqslant 1$$

i.e. $$\left| \sum_{i=1}^{p} \beta(x_i) \, \beta(x_i) \right| \leqslant \| f \| \quad (p \geqslant 1)$$

i.e. $$\sum_{i=1}^{p} | \beta(x_i) | \leqslant \| f \| \qquad (p \geqslant 1)$$

Therefore $\sum_{i=1}^{\infty} | \beta(x_i) | \leqslant \| f \|$. Hence $\| f \| = \| \beta \|$. The uniqueness of f follows easily from the definition of f.

Conversely let $f : A(X, C, s) \to C$ be a metrically bounded linear functional. We define a corresponding $\beta : X \to C$ as $\beta(x) = f(x)$. For $x \in X$ such that $s(x) \geqslant 2$ and for any $c \in C$ we have that

$$| f(c^{s(x)} x) | \leqslant \| f \| \, | c |$$

i.e. $$| c |^{s(x)} \, | f(x) | \leqslant \| f \| \, | c |$$

i.e. $$| \beta(x) | \leqslant \| f \| \, | c |^{1-s(x)}$$

Since $s(x) \geqslant 2$ by letting $| c | \to \infty$ we conclude that $| \beta(x) | = 0$ for every $x \in X$ such that $s(x) \geqslant 2$. Therefore without loss of generality we can assume that $\beta : Y \to C$, and β is such that for any

$$\alpha = \sum_{i=1}^{\infty} \alpha(x_i) . x_i \in A(X, C, s),$$

$$f(\alpha) = \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} \alpha(x_i) \, \beta(x_i).$$

Now to prove that $\beta \in L(Y)$ and $\| \beta \| = \| f \|$, let (x_n) be any sequence in Y, then by defining

$$\alpha_p = \sum_{i=1}^{p} \beta(\hat{x}_i) . x_i \quad \text{for every} \quad p \geqslant 1$$

we conclude that

$$\sum_{i=1}^{p} | \beta(x_i) | \leqslant \| f \| \quad \text{for every} \quad p \geqslant 1.$$

Therefore

$$\sum_{i=1}^{\infty} | \beta(x_i) | \leqslant \| f \|.$$

Hence $\quad \beta \in L(Y)$ and $\| \beta \| \leqslant \| f \|$

For the reverse inequality let

$$\alpha = \sum_{i=1}^{\infty} \alpha(x_i)\, x_i \in A(X, C, s).$$

Then

$$\begin{aligned} | f(\alpha) | &= \Big| \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} \alpha(x_i)\, \beta(x_i) \Big| \\ &\leqslant \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} | \beta(x_i) | \; | \alpha(x_i) | \\ &\leqslant \sum_{\substack{i=1 \\ x_i \in Y}}^{\infty} | \beta(x_i) | \,.\, \sup_{x_i \in Y} | \alpha(x_i) | \\ &\leqslant \| \beta \| \, \| \alpha \|. \end{aligned}$$

This completes the proof, uniqueness being immediate from the definition of the function β.

3. Metrically Bounded Transformations

Definition 3.1 — A linear transformation $T : \Gamma \to \Gamma$ is called a metrically bounded transformation if there exists a positive real number K such that

$$| T(\alpha) | \leqslant K \, | \alpha |$$

for every $\alpha \in \Gamma$

Theorem 3.1 (Ganapathy Iyer) — If $T : \Gamma \to \Gamma$ is a linear transformation such that $| T(\alpha) | \leqslant K \, | \alpha |$ for every $\alpha \in \Gamma$, then

$$T(\delta_0) = a_{00}\delta_0 + a_{01}\delta_1$$

$$T(\delta_1) = a_{10}\delta_0 + a_{11}\delta_1$$

$$T(\delta_n) = k_n\delta_n, \; n \geqslant 2,$$

where a_{ij} are complex numbers such that $| a_{ij} | \leqslant K$ and $| k_n | \leqslant K^n$ for $n \geqslant 2$.

We use Theorem 3.1 to study the stucture of the set $B(\Gamma, \Gamma)$ of all metrically bounded transformations from Γ into Γ. Obviously $B(\Gamma, \Gamma)$ is a vector space with respect to the natural operations. For any $T \in B(\Gamma, \Gamma)$ we define

$$\| T \| = \inf \{ K \mid | T(\alpha) | \leqslant K \, | \alpha | \text{ for every } \alpha \in \Gamma \}. \qquad ...(3.1)$$

It is easily verified that $\| T \|$ defined above satisfies (1.3), (1.4) and (1.5).

Now we consider the set

$$B = \left\{ k = \left(k_n\right)_{n=2}^{\infty} \mid k_n \in C \text{ and there exists } K > 0 \text{ such that } \mid k_n \mid^{1/n} \leqslant K \text{ for every } n \geqslant 2 \right\} \quad \ldots(3.2)$$

B is a vector space with respect to componentwise addition and scalar multiplication. We define, for any $k \in B$,

$$\| k \| = \sup_{n \geqslant 2} \mid k_n \mid^{1/n}, \quad k = \left(k_n\right)_{n=2}^{\infty} \quad \ldots(3.3)$$

It is easy to verify that $\| k \|$ on B satisfies the properties (1.3), (1.4) and (1.5).

Theorem 3.2 — $B(\Gamma, \Gamma) \cong \Gamma^* \oplus \Gamma^* \oplus B$. In other words for each element $\left(f_0, f_1, k = \left(k_n\right)_{n=2}^{\infty}\right)$ of $\Gamma^* \oplus \Gamma^* \oplus B$, there exists $T \in B(\Gamma, \Gamma)$ such that for any $\alpha = \sum_{i=0}^{\infty} a_i \delta_i \in \Gamma$

$$T(\alpha) = f_0(\alpha)\, \delta_0 + f_1(\alpha)\, \delta_1 + \sum_{n=2}^{\infty} a_n k_n \delta_n \quad \ldots(3.4)$$

and in this case

$$\| T \| = \sup_{n \geqslant 2} \{ \| f_0 \|, \| f_1 \|, \mid k_n \mid^{1/n} \} \quad \ldots(3.5)$$

and conversely for any $T \in B(\Gamma, \Gamma)$ there exist $f_0, f_1 \in \Gamma^*$ and a sequence of complex numbers $\left(k_n\right)_{n=2}^{\infty}$ such that for any $\alpha = \sum_{i=0}^{\infty} a_i \delta_i \in \Gamma$

$$T(\alpha) = f_0(\alpha)\, \delta_0 + f_1(\alpha)\, \delta_1 + \sum_{i=2}^{\infty} a_i k_i \delta_i$$

and in this case

$$\| T \| = \sup_{n \geqslant 2} \{ \| f_0 \|, \| f_1 \|, \mid k_n \mid^{1/n} \}.$$

PROOF: Let $\left(f_0, f_1, k = \left(k_n\right)_{n=2}^{\infty}\right)$ be any element of $\Gamma^* \oplus \Gamma^* \oplus B$. Since f_0 and f_1 are elements of Γ^* without loss we can assume $f_0 = (C_{00}, C_{10})$, $f_1 = (C_{01}, C_{11})$ (see Theorem 2.1). Now we define $T : \Gamma \to \Gamma$ as follows:

$$T(\delta_0) = C_{00}\delta_0 + C_{01}\delta_1$$

$$T(\delta_1) = C_{10}\delta_0 + C_{11}\delta_1$$

$$T(\delta_n) = k_n \delta_n, \; n \geqslant 2.$$

To prove that the above defined T satisfies (3.4) and (3.5) we let $\alpha = \sum_{i=0}^{\infty} a_i\delta_i \in \Gamma$ then

$$T(\alpha) = \sum_{i=0}^{\infty} a_i T(\delta_i)$$

$$= f_0(\alpha)\,\delta_0 + f_1(\alpha)\,\delta_1 + \sum_{i=2}^{\infty} a_i k_i \delta_i$$

This proves (3.4). To prove (3.5) we proceed as follows:

$$| T(\alpha) | = \left| f_0(\alpha)\,\delta_0 + f_1(\alpha)\,\delta_1 + \sum_{i=2}^{\infty} a_i k_i \delta_i \right|$$

$$= \sup_{n \geqslant 2} \{ | f_0(\alpha) | , | f_1(\alpha) | , | a_n k_n |^{1/n} \}$$

$$\leqslant \sup_{n \geqslant 2} \{ \| f_0 \| \, | \alpha | , \| f_1 \| \, | \alpha | , | k_n |^{1/n} \}$$

$$| T(\alpha) | \leqslant \sup_{n \geqslant 2} \{ \| f_0 \|, \| f_1 \|, | k_n |^{1/n} \} \, | \alpha | . \quad ...(3.6)$$

Since $\left(f_0, f_1, k = \left(k_n \right)_{n=2}^{\infty} \right)$ is an element of $\Gamma^* \oplus \Gamma^* \oplus B$ we have that the set $\{ \| f_0 \|, \| f_1 \|, | k_n |^{1/n} \mid n \geqslant 2 \}$ is bounded and hence the supremum in (3.6) exists and we conclude that

$$\| T \| \leqslant \sup_{n \geqslant 2} \{ \| f_0 \|, \| f_1 \|, | k_n |^{1/n} \}. \quad ...(3.7)$$

To prove (3.5) we need to prove the reverse inequality and we prove this in the following three parts:

$$\| f_0 \| \leqslant \| T \| \quad ...(3.8)$$

$$\| f_1 \| \leqslant \| T \| \quad ...(3.9)$$

$$| k_n |^{1/n} \leqslant \| T \|. \quad ...(3.10)$$

Let $\beta_1 \in \Gamma$ be defined as $\beta_1 = \hat{C}_{00}\delta_0 + \hat{C}_{10}\delta_1$. Since by (3.6) T is a metrically bounded transformation and $| \beta_1 | \leqslant 1$ we have that

$$| T(\beta_1) | \leqslant \| T \| \, | \beta_1 | \leqslant \| T \|,$$

i.e. $| \hat{C}_{00} C_{00} + \hat{C}_{10} C_{10} \leqslant \| T \|,$

i.e. $| C_{00} | + | C_{10} | \leqslant \| T \|,$

i.e. $\| f_0 \| \leqslant \| T \|.$

(3.9) follows similarly as (3.8) where we define $\beta_2 \in \Gamma$ as $\beta_2 = \hat{C}_{01}\delta_0 + \hat{C}_{11}\delta_1$. Since T is metrically bounded we have for any $n \geqslant 2$, that

$$| T(\delta_n) | \leqslant \| T \|.$$

i.e. $$| k_n \delta_n | \leqslant \| T \|.$$

i.e. $$| k_n |^{1/n} \leqslant \| T \|$$

which is nothing but (3.10). In view of (3.8), (3.9) and (3.10) we have

$$\sup_{n \geqslant 2} \{ \| f_0 \|, \| f_1 \|, | k_n |^{1/n} \} \leqslant \| T \|. \quad \ldots(3.11)$$

(3.5) is now an easy consequence of (3.7) and (3.11)

Conversely let $T : \Gamma \rightarrow \Gamma$ be a metrically bounded transformation, then in view of Theorem 3.1, we have that

$$T(\delta_0) = a_{00}\delta_0 + a_{01}\delta_1$$

$$T(\delta_1) = a_{10}\delta_0 + a_{11}\delta_1$$

$$T(\delta_n) = k_n \delta_n, \ n \geqslant 2$$

where a_{ij} and k_n, $n \geqslant 2$ are complex numbers such that $| a_{ij} | \leqslant K$ and $| k_n | \leqslant K^n$ for some positive real number K. We now define

$$f_0 = (a_{00}, a_{10}) \in \Gamma^*$$

$$f_1 = (a_{01}, a_{11}) \in \Gamma^*$$

and $k = \left(k_n\right)_{n=2}^{\infty} \in B.$

Then by an argument similar to that in first part of the theorem we can easily prove that $\left(f_0, f_1, k = \left(k_n\right)_{n=2}^{\infty} \right)$ is an element of $\Gamma^* \oplus \Gamma^* \oplus B$ and that (3.4), and (3.5) are satisfied. This completes the proof of the theorem.

Theorem 3.3 — The $\| T \|$ defined in Theorem 3.2 on the space $B(\Gamma, \Gamma)$ satisfies the following properties:

$$\| T \| \geqslant 0 \text{ and } \| T \| = 0 \text{ if and only if } T = 0 \quad \ldots(3.12)$$

$$\| T + S \| \leqslant \| T \| + \| S \| \quad \ldots(3.13)$$

$$\| \lambda T \| \leqslant A(\lambda) \| T \| \quad \ldots(3.14)$$

where $\lambda \in C$ and $A(\lambda) = \max \{1, | \lambda | \}$.

If $\lambda_q \rightarrow \lambda$, then it is not true in general that

$$\lambda_q T \rightarrow \lambda T \text{ for } T \in B(\Gamma, \Gamma). \quad \ldots(3.15)$$

PROOF : (3.12), (3.13) and (3.14) are immediate consequences of the definition of $\| T \|$. To prove (3.15) it is sufficient to give an example of a sequence $\left(c_n\right)_{n=1}^{\infty}$ and an element $T \in B(\Gamma, \Gamma)$ such that $c_n \to 0$ but that $c_n T$ does not tend to zero.

Let $c_n = 2^{-n}$ for $n \geqslant 2$ and $T = (0, 0, k_n = 1, n \geqslant 2) \in B(\Gamma, \Gamma)$ then we have that $c_n \to 0$ but that

$$\| c_n T \| = \sup_{p \geqslant 2} (2^{-n})^{1/p} \geqslant \tfrac{1}{2}.$$

This means $\| c_n T \| \geqslant \frac{1}{2}$ for all n and hence $c_n T$ does not tend to zero.

The proof of the following corollary is straightforward.

Corollary 3.1 — If T_n, T, S are elements of $B(\Gamma, \Gamma)$ then:

If $\quad T_n \to T$, then $cT_n \to cT (c \in C)$ ...(3.16)

$T + S \in B(\Gamma, \Gamma)$ ...(3.17)

Remark 3.1 : $B(\Gamma, \Gamma)$ is a group under addition. (3.17) shows that $B(\Gamma, \Gamma)$ is in fact a topological group. But $B(\Gamma, \Gamma)$ is not a metric linear space like Γ since the scalar multiplication is not continuous in the variable c.

Theorem 3.4 — $\Gamma^* \oplus \Gamma^* \oplus L(\delta_n \mid n \geqslant 2)$ is the greatest subset of $B(\Gamma, \Gamma)$ which is at the same time a linear metric space i.e. if Φ is a linear subspace of $B(\Gamma, \Gamma)$ and Φ, regarded as metric subspace of $B(\Gamma, \Gamma)$, is also a metric linear space then $\Phi \subset \Gamma^* \oplus \Gamma^* \oplus L(\delta_n \mid n \geqslant 2)$, where by $L(\delta_n \mid n \geqslant 2)$ we mean the linear subspace of Γ spanned by δ_n, $n \geqslant 2$.

PROOF : $\Gamma^* \oplus \Gamma^* \oplus L(\delta_n \mid n \geqslant 2)$ is a metric linear subspace of $B(\Gamma, \Gamma)$, since Γ^* is a Banach space and $L(\delta_n \mid n \geqslant 2)$ is a metric linear subspace of Γ. In view of Theorem 3.2 we have that $\Gamma^* \oplus \Gamma^* \oplus L(\delta_n \mid n \geqslant 2) \subset B(\Gamma, \Gamma)$, and is also a metric subspace of $B(\Gamma, \Gamma)$.

Let Φ be a metric linear subspace of $B(\Gamma, \Gamma)$. We need to prove that

$$\Phi \subset \Gamma^* \oplus \Gamma^* \oplus L(\delta_n \mid n \geqslant 2).$$

To prove this we assume the contrary. Hence let if possible T be an element of $B(\Gamma, \Gamma)$ such that $T \in \Phi$ but $T \notin \Gamma^* \oplus \Gamma^* \oplus L(\delta_n \mid n \geqslant 2)$. This is possible only if $| k_n |^{1/n}$ does not tend to zero, where by Theorem 3.2 we can assume

$$T = (f_0, f_1, k = (k_n)_{n \geqslant 2}).$$

Hence we can find a real number $\eta > 0$ such that $| k_n |^{1/n} \geqslant \eta$ for every $n \geqslant 2$. Now we define

$$C_n = \frac{1}{(1+\eta)^{n^2}}, \; n \geqslant 2.$$

Then clearly $C_n \to 0$ and

$$\| C_p T \| = \sup_{n \geqslant 2} \{ | C_p | \, \| f_0 \|, \, | C_p | \, \| f_1 \|, \, | C_p |^{1/n} \, | k_n |^{1/n} \}$$

$$\geqslant \sup_{n \geqslant 2} \left\{ | C_p | \, \| f_0 \|, \, | C_p | \, \| f_1 \|, \, \frac{\eta}{(1+\eta)^{p/n}} \right\}$$

$$\geqslant \frac{\eta}{1+\eta}$$

Hence we have proved that $\| C_p T \| \geqslant \frac{\eta}{1+\eta}$ for every $p \geqslant 2$, consequently $c_n T$ does not tend to zero, which is a contradiction to the fact that Φ is metric linear. This completes the proof.

Theorem 3.5 — If $T_n \to T$ in $B(\Gamma, \Gamma)$ then $T_n(\alpha) \to T(\alpha)$ for every $\alpha \in \Gamma$ and this convergence is uniform on the unit disk in Γ.

PROOF : $| (T_n - T) \alpha | \leqslant \| T_n - T \| \, | \alpha |$

and $| (T_n - T) \alpha | \leqslant \| T_n - T \|$ if $| \alpha | \leqslant 1$.

The proof is now immediate.

Let $\triangle$ denote the set of all metrically bounded transformations of $A(X, C, s)$ to $A(X, C, s)$.

Note 3.1: We consider the set $C_0(X) = \{f : X \to C \mid$ for every $\epsilon > 0$ there exist only finitely many $x \in X$ such that $| f(x) | \geqslant \epsilon\}$. It is easily verified that $c_0(X)$ is a Banach space with respect to the norm defined on it as

$$\| f \| = \sup_{x \in X} | f(x) | . \qquad ...(3.18)$$

We further observe that each element $f \in c_0(X)$ can be represented by $(f_{x_i})_{i \geqslant 1}$ for some sequence (x_n) in X, since the set $\{x \in X \mid f(x) \neq 0\}$ is countable.

Note 3.2 : Let B be any Banach space. We define $L(X, B)$ as follows:

$$L(X, B) = \Big\{ f : X \to B \mid \text{there exists } K > 0 \text{ such that for any sequence } (x_n) \text{ in } X \sum_{n=1}^{\infty} \| f(x_n) \| \leqslant K \Big\}. \qquad ...(3.19)$$

For any $f \in L(X, B)$ we then define

$$\| f \| = \sup_{(x_n) \subset X} \sum_{n=1}^{\infty} \| f(x_n) \|. \qquad ...(3.20)$$

$L(X, B)$ becomes a Banach space with respect to the norm $\| f \|$.

Theorem 3.6 — If T is a metrically bounded transformation from $A(X, C, s)$ to $A(X, C, s)$, then for each $x \in X$ there is an associated sequence (x_i) in X and a sequence (a_{xx_i}) in C depending on x and x_i such that

$$T(x) = \sum_{\substack{i=1 \\ t(x_i)=t(x)}}^{\infty} a_{xx_i} x_i \qquad \ldots(3.21)$$

$$| a_{xx_i} | \leqslant K^{1/t(x)} \text{ for all } i \qquad \ldots(3.22)$$

$$| a_{xx_i} | \to 0 \text{ as } 0 \to \infty. \qquad \ldots(3.23)$$

PROOF : For every $x \in X$, $T(x)$ being an element of $A(X, C, s)$, in view of (1.6) we can assume that

$$T(x) = \sum_{i=1}^{\infty} a_{xx_i} x_i$$

Let $c \in C$, then

$$\| T(c^{1/t(x)} . x) \| \leqslant K | c | ,$$

i.e.

$$\sup_{i \geqslant 1} | c |^{t(x_i)/t(x)} | a_{xx_i} |^{t(x_i)} \leqslant K | c |$$

i.e.

$$| a_{xx_i} | \leqslant K | c |^{(t(x)-t(x_i))/t(x)}, \; (i \geqslant 1). \qquad \ldots(3.24)$$

Therefore by letting $| c | \to 0$ or ∞ as required we conclude that

$$| a_{xx_i} | = 0 \text{ for any } i \text{ such that } t(x_i) \neq t(x). \quad \text{Hence}$$

$$T(x) = \sum_{\substack{i=1 \\ t(x_i)=t(x)}}^{\infty} a_{xx_i} . x_i$$

Hence we have proved (3.21). (3.22) follows easily by substituting $t(x) = t(x_i)$ in (3.24) and (3.23) follows from the fact that $T(x) \in A(X, C, s)$.

In view of the Theorem 3.6 above we can view a sequence (a_{xx_i}) as an element of $c_0\left(t^{-1}(t(x)) \right)$. We then have

$$\| (a_{xx_i}) \| \leqslant K^{1/t(x)}.$$

Theorem 3.7 — Let T be a metrically bounded transformation from $A(X, C, s)$ into $A(X, C, s)$. For each $p \geqslant 1$ we define

$$T_p : t^{-1}(1/p) \to c_0(t^{-1}(1/p))$$
$$x \to (a_{xx_i})$$

Then $$T_p \in L\left(t^{-1}(1/p), c_0(t^{-1}(1/p))\right)$$

and

$$\| T_p \| \leqslant K^p.$$

PROOF: Firstly we claim that for any $\upsilon \in t^{-1}(1/p)$ and any sequence (x_n) in $t^{-1}(1/p)$, $\sum_{n=1}^{\infty} | a_{x_n \upsilon} | \leqslant K^p$. For this let (x_n) be any sequence in X such that $t(x_n) = 1/p$ for every $n \geqslant 1$. We define for any $m \geqslant 1$, β_m as follows:

$$\beta_m : X \to C$$

$$y \to 0 \text{ if } y \neq x_n, 1 \leqslant n \leqslant m$$

$$x_n \to \hat{a}_{x_n \upsilon}.$$

Without loss of generality we can assume that $\| \beta_m \| = 1$, so that

$$\| T(\beta_m) \| \leqslant K$$

i.e. $$\left\| T\left(\sum_{n=1}^{m} \hat{a}_{x_n \upsilon} . x_n\right)\right\| \leqslant K$$

i.e. $$\left\| \sum_{n=1}^{m} \hat{a}_{x_n \upsilon} T(x_n) \right\| \leqslant K$$

i.e. $$\left\| \sum_{n=1}^{m} \hat{a}_{x_n \upsilon} \sum_{i=1}^{\infty} a_{x_n x_{n_i}} x_{n_i} \right\| \leqslant K$$

i.e. $$\left\| \sum_{v=1}^{\infty} \left(\sum_{n=1}^{m} \hat{a}_{x_n \upsilon} a_{x_n x_v} \right) x_v \right\| \leqslant K$$

i.e. $$\sup_{v \geqslant 1} \left| \sum_{n=1}^{m} \hat{a}_{x_n \upsilon} a_{x_n x_v} \right|^{t(x_v)} \leqslant K$$

i.e. $$\sup_{v \geqslant 1} \left| \sum_{n=1}^{m} \hat{a}_{x_n \upsilon} a_{x_n x_v} \right|^{1/p} \leqslant K.$$

In particular when $x_v = \upsilon$ we have that

$$\sum_{n=1}^{m} | a_{x_n} | \leqslant K^p.$$

Since this is true for every $m \geqslant 1$

we have that

$$\sum_{n=1}^{\infty} | a_{x_n} | \leqslant K^p. \qquad \text{...(3.25)}$$

Hence we have proved our claim.

Now we prove that, for any sequence (x_n) in $t^{-1}(1/p)$,

$$\sum_{n=1}^{\infty} \| (a_{x_n x_{n_i}}) \| \leqslant K^p.$$

For this we assume on the contrary that

$$\sum_{n=1}^{\infty} \| (a_{x_n x_{n_i}}) \| > K^p.$$

Then there exists N such that

$$\sum_{n=1}^{N} \| (a_{x_n x_{n_i}}) \| > K^p.$$

i.e.
$$\sum_{n=1}^{N} \sup_{i \geqslant 1} | a_{x_n x_{n_i}} | > K^p$$

i.e.
$$\sup_{\substack{n_i \geqslant 1 \\ 1 \leqslant n \leqslant N}} \sum_{n=1}^{N} | a_{x_n x_{n_i}} | > K^p.$$

Therefore there exists $n_i = \upsilon$ (say) such that

$$\sum_{n=1}^{N} | a_{x_n \upsilon} | > K^p.$$

This being a contradiction to our claim (3.25) we are through.

Δ, the set of all metrically bounded linear transformations of $A(X, C, s)$ to $A(X, C, s)$ is clearly a vector space with respect to natural operations. We define for any $T \in \Delta$

$$\| T \| = \inf \{K \mid \| T(\alpha) \| \leqslant K \| \alpha \| \text{ for every } \alpha \in A(X, C, s)\}. \quad \ldots(3.26)$$

Then clearly

$\| T(\alpha) \| \leqslant \| T \| \, \| \alpha \|$ for every $\alpha \in A(X, C, s)$, and $\| T \|$ satisfies (1.3), (1.4) and (1.5).

Let for every $n \geqslant 0$, B_n denote a Banach space over C. Let

$$D \langle B_n, n \geqslant 0 \rangle = \left\{ \sum_{n=0}^{\infty} b_n \delta_n \mid b_n \in B_n \text{ for every } n \geqslant 1 \text{ and } \| b_n \|^{1/n} \to 0 \text{ as } n \to \infty \right\}. \quad \ldots(3.27)$$

Then it is easily verified, with the help of arguments similar to that for Γ, that $D \langle B_n, n \geqslant 0 \rangle$ is a complete metric linear space with respect to metric arising from

$$\left\| \sum_{n=0}^{\infty} b_n \delta_n \right\| = \sup_{n \geqslant 1} (\| b_0 \|, \| b_n \|^{1/n}) \qquad ...(3.28)$$

which satisfies (1.3), (1.4) and (1.5). We consider the vector space $\widetilde{D} \langle B_n, n \geqslant 0 \rangle$ as follows

$$\widetilde{D} \langle B_n, n \geqslant 0 \rangle = \Big\{ \sum_{n=0}^{\infty} b_n \delta_n \mid b_n \in B_n \text{ for every } n \geqslant 0 \text{ and there exists } K \geqslant 0 \text{ such that } \| b_n \|^{1/n} \leqslant K \text{ for all } n \geqslant 1 \Big\}. \qquad ...(3.29)$$

Clearly $D \langle B_n, n \geqslant 0 \rangle$ is the unique maximal metric linear subspace of

$$\widetilde{D} \langle B_n, n \geqslant 0 \rangle,$$

where $\left\| \sum_{n=0}^{\infty} b_n \delta_n \right\|$ is defined on $\widetilde{D} \langle B_n, n \geqslant 0 \rangle$ as in (3.28) and it satisfies (3.12), (3.13), (3.14) and (3.15).

Following theorem is an easy consequence of Theorems 3.6 and 3.7.

Theorem 3.8 — For any metrically bounded transformation T from $A(X, C, s)$ into $A(X, C, s)$,

$$\sum_{p=1}^{\infty} T_p \delta_p \in \widetilde{D} \Big\langle L\Big(t^{-1}(1/p), c_0(t^{-1}(1/p))\Big), \ p \geqslant 1 \Big\rangle$$

and
$$\left\| \sum_{p=1}^{\infty} T_p \delta_p \right\| \leqslant \| T \|.$$

Theorem 3.9 — $\| T \| = \left\| \sum_{p=1}^{\infty} T_p \delta_p \right\|.$

PROOF: In view of the Theorem 3.8 we have only to prove

$$\| T \| \leqslant \left\| \sum_{p=1}^{\infty} T_p \delta_p \right\|.$$

For this let $\alpha = \sum_{j=1}^{\infty} \alpha(x_j) \,.\, x_j \in A(X, C, s)$. Then

$$\begin{aligned} \| T(\alpha) \| &= \left\| \sum_{j=1}^{\infty} \alpha(x_j) \, T(x_j) \right\| \\ &= \left\| \sum_{j=1}^{\infty} \alpha(x_j) \sum_{i=1}^{\infty} a_{x_j x_{ji}} \, x_{ji} \right\| \\ &= \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \alpha(x_j) \, a_{x_j x_{ji}} \,.\, x_{ji} \right\| \end{aligned}$$

(*equation continued on p.* 219)

$$= \left\| \sum_{v=1}^{\infty} \left(\sum_{y_v = x_{ji}} \alpha(x_j)\, a_{x_j x_{ji}} \right) y_v \right\|$$

$$= \sup_{v \geqslant 1} \left| \sum_{y_v = x_{ji}} \alpha(x_j)\, a_{x_j x_{ji}} \right|^{t(x_{ji})},$$

but $t(x_{ji}) = t(x_j) = t(y_v)$. Therefore

$$\| T(\alpha) \| \leqslant \sup_{v \geqslant 1} \left[\sum_{y_v = x_{ji}} \left| \alpha(x_j) \right| \left| a_{x_j x_{ji}} \right| \right]^{t(x_j)}.$$

In view of Note 3.1, we have that

$$\| T(\alpha) \| \leqslant \sup_{v \geqslant 1} \left[\sum_{y_v = x_{ji}} \left| \alpha(x_j) \right| \left\| (a_{x_j x_{ji}}) \right\| \right]^{t(x_j)}$$

$$\leqslant \sup_{v \geqslant 1} \left[\| \alpha \|^{1/t(x_j)} \sum_{y_v = x_{ji}} \left\| (a_{x_j x_{ji}}) \right\| \right]^{t(x_j)}$$

and in view of (3.20),

$$\| T(\alpha) \| \leqslant \sup_{j \geqslant 1} \| \alpha \| \, . \, \| T_{1/t(x_j)} \|^{t(x_j)},$$

$$\leqslant \left\| \sum_{p=1}^{\infty} T_p \delta_p \right\| . \| \alpha \|.$$

This follows from (3.28). This completes the proof of the Theorem.

Theorem 3.10 — $\Delta = \widetilde{D} \left\langle L\left(t^{-1}(1/p), c_0(t^{-1}(1/p))\right), p \geqslant 1 \right\rangle$ ...(3.30)

PROOF : Consider the map

$$\theta : \Delta \to \widetilde{D} \left\langle L\left(t^{-1}(1/p), c_0(t^{-1}(1/p))\right), p \geqslant 1 \right\rangle$$

$$T \to \sum_{p=1}^{\infty} T_p \delta_p.$$

Now by Theorems 3.6, 3.7, 3.8 and (3.9) θ is one-one and preserves $\| . \|$.

It remains to prove that θ is onto. For this we prove the following:

If $\sum_{p=1}^{\infty} b_p \delta_p$ is an element in right-hand side of (3.30). Then for any $p \geqslant 1$

$$b_p \in L\left(t^{-1}(1/p), c_0(t^{-1}(1/p))\right)$$

i.e.

$$b_p : t^{-1}(1/p) \to c_0(t^{-1}(1/p))$$

$$x \to (b_{p_{x x_i}}).$$

Now we define a linear transformation T as

$$T(x) = \sum_{i=1}^{\infty} b_{p_{xx_i}} \cdot x_i$$

Firstly we prove that for any $\alpha = \sum_{j=1}^{\infty} \alpha(x_j) \cdot x_j \in A(X, C, s)$ $T(\alpha)$ exists. But

$$T(\alpha) = \sum_{j=1}^{\infty} \alpha(x_j)\, T(x_j)$$

$$= \sum_{j=1}^{\infty} \alpha(x_j) \sum_{i=1}^{\infty} b_{(1/t(x_j))_{x_j x_{ji}}}\, x_{ji}$$

$$= \sum_{i.j=1}^{\infty} \Bigg(\sum_{\substack{v=1 \\ y_v = x_{ji}}}^{\infty} b_{(1/t(x_j))_{x_j y_v}} \cdot \alpha(x_j) \Bigg)\, x_{ji}.$$

To prove that $T(\alpha) \in A(X, C, s)$, we need to prove that

$$\lim_{i,j\to\infty} \Bigg| \sum_{\substack{v=1 \\ y_v = x_{ji}}}^{\infty} b_{(1/t(x_j))_{x_j y_v}} \cdot \alpha(x_j) \Bigg|^{t(x_j)} = 0 .$$

But

$$\lim_{t,j\to\infty} \Bigg| \sum_{\substack{v=1 \\ y_v = x_{ji}}}^{\infty} b_{(1/t(x_j))_{x_j y_v}} \cdot \alpha(x_j) \Bigg|^{t(x_j)}$$

$$\leqslant \lim_{ij\to\infty} \Bigg[\sum_{\substack{v=1 \\ y_v = x_{ji}}}^{\infty} \Big| b_{(1/t(x_j))_{x_j y_v}} \Big| \Bigg]^{t(x_j)} | \alpha(x_j) |^{t(x_j)}$$

$$\leqslant \lim_{j\to\infty} \| b_{(1/t(x_j))} \|^{t(x_j)} \, | \alpha(x_j) |^{t(x_j)}$$

$$\leqslant \lim_{j\to\infty} K\, | \alpha(x_j) |^{t(x_j)}$$

$$= 0.$$

Hence $T(\alpha) \in A(X, C, s)$. Since a convergent series can be rearranged without altering its sum we can prove that $\theta(T) = \sum_{p=1}^{\infty} b_p \delta_p$, proceeding similarly as in Theorems 3.6, 3.7, 3.8, and 3.9. This completes the proof.

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