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NOTE ON KONHAUSER BIORTHOGONAL POLYNOMIALS

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In this paper generating functions for $Y_n^\alpha(x; k)$ and the double integral involving $Z_n^\alpha(x; k)$ and $Y_n^\alpha(x; k)$ is obtained.

Konhauser (1965, 1967) has discussed two polynomials

$$Z_n^\alpha(x; k) = \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)}. \qquad \ldots(1)$$

$$Y_n^\alpha(x; k) = \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left[\frac{e^{-xt}(1+t)^{\alpha+kn}}{(t^{k-1}+kt^{k-2}+\ldots+k)^{n+1}} \right]\Bigg|_{t=0} \qquad \ldots(2)$$

biorthogonal with respect to the weight function $x^\alpha \exp(-x)$ over the interval $(0, \infty)$.

Recently Carlitz (1968), Prabhakar (1970, 1971), Srivastava (1973), Srivastava and Singh (1979), Karande and Thakare (1975) and Patil and Thakare (1978) have studied these polynomials. For $Y_n^\alpha(x; k)$ Patil (1977) has given the operational formula

$$Y_n^{\alpha+\lambda}(x; k) = \frac{x^{-(\alpha+1+kn)}}{k^n n!} \exp(x)\, \theta^n \{\exp(-x)\, x^{\alpha+1}\} \qquad \ldots(3)$$

where $\theta = \lambda x^k + x^{k+1}D$, $D \equiv \frac{d}{dx}$.

The purpose of this paper is to obtain some generating functions and an integral involving Konhauser biorthogonal polynomials.

GENERATING FUNCTION

Using (1) we write

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha+\lambda}(x; k)\, t^n = \frac{e^{-x} x^{-(\alpha+1+km)}}{k^m m!} \sum_{n=0}^{\infty} \frac{x^{-kn} t^n}{k^n n!} \theta^n \theta^m (e^{-x} x^{\alpha+1})$$

(*equation continued on p.* 223)

$$= e^{-x}x^{-(\alpha+1+kn)} \exp\left(\frac{t\theta}{kx^k}\right)\left(x^{\alpha+1+km}e^{-x}Y_m^{\alpha+\lambda}(x;k)\right)$$

$$= \{1-t\}^{-(\alpha+1+\lambda+km)/k} \exp\left[x\{1-(1-t)^{-1/k}\}\right]$$

$$\times\, Y_m^{\alpha+\lambda}\{x(1-t)^{-1/k};k\}. \qquad \ldots(4)$$

Here we have used the property of the operator θ

$$e^{t\theta}(x^{\alpha}\,.\,f(x)) = \frac{x^{\alpha}}{(1-tkx^k)^{(\alpha+\lambda)/k}}\, f\left(\frac{x}{(1-tkx^k)^{1/k}}\right). \qquad \ldots(5)$$

Now consider the expansion

$$x^{\alpha+1}e^{-x}e^{xt} = \sum_{r=0}^{\infty} e^{-x}x^{\alpha+1+r}\,\frac{t^r}{r!}.$$

Operating on both sides with θ^n we obtain the generating function

$$\sum_{r=0}^{\infty} Y_n^{\alpha+\lambda+r}(x;k)\,\frac{t^r}{r!} = x^{-\lambda}e^{-t}\,Y_n^{\alpha+\lambda}(x-t;k) \qquad \ldots(6)$$

which seems to be new.

As a special case $\lambda = 0$, $k = 1$ we get the generating function (Buchholtz 1953, pp. 142)

$$e^t L_n^{\alpha}(x-t) = \sum_{r=0}^{\infty} L_n^{\alpha+r}(x)\,\frac{t^r}{r!}. \qquad \ldots(7)$$

Next for $Y_n^{\alpha}(x;k)$ Carlitz (1968) has given the generating function

$$\sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\,t^n = (1-t)^{-(\alpha+1)/k} \exp\left[-x\{(1-t)^{-1/k}-1\}\right]. \qquad \ldots(8)$$

By Taylor's theorem we write

$$Y_n^{\alpha}(x;k) = \frac{1}{n!}\,D_t^n\left[(1-t)^{-(\alpha+1)/k} \exp\{-x((1-t)^{-1/k}-1)\}\right]_{t=0} \qquad \ldots(9)$$

where $D_t \equiv \dfrac{d}{dt}$.

Hence

$$Y_n^{\alpha+kbn}(x;k) = \frac{1}{n!} D_t^n \left[(1-t)^{-(\alpha+1)/k} \exp\{-x\{(1-t)^{-1/k}-1\}\} \times (1-t)^{-bn}\right]_{t=0} \quad \ldots(10)$$

where b is any complex number.

Now by using Lagrange's theorem (Polya and Szegö 1925, pp. 125)

$$\frac{f(z)}{1-u\phi'(z)} = \sum_{n=0}^{\infty} \frac{u^n}{n!} \left[D^n \{f(x)\,(\phi(x))^n\}\right]_{x=0} \quad \ldots(11)$$

where $u = z/\phi(z)$, in (10) we obtain the generating function

$$\sum_{n=0}^{\infty} Y_n^{\alpha+kbn}(x;k)\, u^n = \frac{(1+v)^{(\alpha+1)/k} \exp[-x\{1+v)^{1/k}-1\}]}{1-bv} \quad \ldots(12)$$

with $u = v(1+v)^{-b-1}$.

In particular for $b = -1$ we get the generating function obtained by Srivastava and Singh (1979)

Double Integral

Edwards (1922, pp. 177) has given the relation

$$\int_0^{\infty}\int_0^{\infty} f(x+y)\, x^{\alpha} y^{\beta}\, dx\, dy$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \int_0^{\infty} f(z)\, z^{\alpha+\beta+1}\, dz. \quad \ldots(13)$$

Let $f(x+y) = e^{-(x+y)} Z_n^{\alpha+\beta+1}(x+y;k)\, Y_m^{\alpha+\beta+1}(x+y;k).$

By using the biorthogonal property

$$\int_0^{\infty} e^{-x} x^{\alpha} Z_n^{\alpha}(x;k)\, Y_m^{\alpha}(x;k)\, dx = 0 \qquad \text{if } m \neq n$$

$$= \frac{\Gamma(kn+\alpha+1)}{n!} \quad \text{if } m = n \quad \ldots(14)$$

we obtain the required integral

$$\int_0^\infty \int_0^\infty e^{-(x+y)} x^\alpha y^\beta Z_n^{\alpha+\beta+1}(x+y;k)\, Y_m^{\alpha+\beta+1}(x+y;k)\, dx\, dy$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,(\alpha+\beta+2)_{kn}}{n!} \quad \text{if} \quad m = n$$

$$= 0 \quad \text{if} \quad m \neq n. \qquad \ldots(15)$$

For $k = 1$, we get the results for Laguerre polynomials.

REFERENCES

Buchholtz, H. (1953). Die Konfluente Hypergeometrische Funktionen. Berlin.

Carlitz, L. (1968). A note on certain biorthogonal polynomials. *Pacific J. Math.*, **24**, 425–30.

Edwards, J. (1922). A Treatise of the Integral Calculus, Vol. 11.

Polya, G., and Szegö, G. (1925). Aufgaben und Lehrsatze aus der Analysis, I. Springer-Verlag, Berlin.

Karande, B. K., and Thakare, N. K. (1975). On polynomials related to Konhauser biorthogonal polynomials. *Math. Student*, **43**, 67–72.

Konhauser, J. D. E. (1965). Some properties of biorthogonal polynomials. *J. Math. Anal. Appl.*, **11**, No. 1–3, 242–60.

——— (1967). Biorthogonal polynomials suggested by the Laguerre polynomials. *Pacific J. Math.*, **21**, 303–14.

Patil, K. R. (1977). Some problems connected with higher transcendental functions. Ph.D. Thesis.

Patil, K. R., and Thakare, N. K. (1978). Multilinear generating function for the Konhauser biorthogonal polynomial sets. *SIAM J. Math. Anal.*, **9**, 921–23.

Prabhakar, T. R. (1970). On a set of polynomials suggested by Laguerre polynomials. *Pacific J. Math.*, **35**, 213–19.

——— (1971). On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials. *Pacific J. Math.*, **37**, 801–804.

Srivastava, H. M. (1973). On the Konhauser set of biorthogonal polynomials suggested by the Laguerre polynomials. *Pacific J. Math.*, **39**, 489–92.

Srivastava, A. N., and Singh, S. N. (1979). On the Konhauser polynomials $Y_n^\alpha(x;k)$. *Indian J. pure appl. Math.*, **10**, 1121–26.