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WAVE PROPAGATION IN A THERMOELASTIC HALF-SPACE

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One-dimensional dynamical disturbances in a thermoelastic half-space with plane boundary due to the application of a step in strain or temperature on the boundary are studied in the context of the linearized Green-Lindsay (1972) thermoelasticity theory. The solution is obtained by the use of integral transforms. Short and long time approximations of solutions are deduced and the exact discontinuities in the mechanical and thermal fields are analysed. Some of the earlier results are deduced as particular cases of the more general results obtained here.

1. INTRODUCTION

The disturbances produced in a thermoelastic half-space due to the application of time-dependent loading or heating to the boundary have been studied by several authors; vide, for example, Boley and Tolins (1962), Lord and Shulman (1967), Popov (1967), Norwood and Warren (1969) and Johnson (1975). Boley and Tolins (1962) have employed the field equations of the linear coupled thermoelasticity theory to study the disturbances and in the studies by Lord and Shulman (1967), Popov (1967) and Norwood and Warren (1969) the linearized equations of a more general theory based on a modified law of heat conduction have been employed. Both linear and second-order equations of the coupled theory have been employed and a comprehensive comparitive study of the results obtained in the two cases has been made by Johnson (1975).

The purpose of the present paper is to reinvestigate the problems considered by Boley and Tolins (1962), Lord and Shulman (1967), Popov (1967) and Norwood and Warren (1969) in the context of the linearized Green-Lindsay (1972) thermoelasticity theory. Like the Lord-Shulman theory, the Green-Lindsay theory is also a generalization of the coupled theory, which allows for so-called "second-sound" effects. But there exist the following differences between the two theories.

(i) The Lord-Shulman theory modifies only the energy equation of the coupled theory to take into account the time needed for the acceleration of the heat flow. The Green-Lindsay theory modifies both the constitutive equation and the energy equation. Accordingly, the Lord-Shulman theory involves only one relaxation time of the thermoelastic process and the Green-Lindsay theory involves two relaxation times.

(ii) The energy equation of the Lord-Shulman theory depends both on the strain-velocity and strain-acceleration whereas the corresponding equation of the Green-Lindsay theory depends only on the strain-velocity.

(iii) In the linearized case, according to the Green-Lindsay theory the heat cannot propagate with a finite speed unless the stresses depend on the temperature-velocity. According to the Lord-Shulman theory the heat can propagate with a finite speed eventhough the stresses there are independent of the temperature-velocity.

Further, it may be remarked that some uniqueness theorems and a domain of influence theorem have been established in the context of the Green-Lindsay theory (Green and Lindsay 1972; Green 1972; Ignaczak 1978a, b), while no such theorems have been obtained in the context of the Lord-Shulman theory.

In view of the differences between the Green-Lindsay theory and the coupled and Lord-Shulman theories outlined above, some of the results obtained in this paper differ from those obtained by Boley and Tolins, Lord and Shulman, Popov, and Norwood and Warren. However, in the special case when the two relaxation times are equal, some of our results reduce to those obtained by Lord and Shulman, Popov, and Norwood and Warren. In the limiting case when the relaxation times tend to zero, all our results reduce to those obtained by Boley and Tolins.

In section 2 we state the basic equations of the linearized Green-Lindsay theory and reduce these equations to dimensionless form. In section 3 we formulate the governing equations and the initial and boundary conditions for the one-dimensional deformation of a half-space, stating the assumptions made. We obtain the solution in transforms by the use of the Laplace transform on time and the sine and cosine transforms on space. Since the "second-sound" effects are small and short lived (Green and Lindsay 1972), we consider the short time approximation of the solution and find that the solution in general represents two waves which propagate with finite speeds. One of these is the acoustic wave influenced by the thermal field and the other is the thermal wave, and the latter wave precedes the former one. Further, the acoustic wave is slower than its counterpart in the Lord-Shulman theory.

In sections 4-6 we consider specific problems which are particular cases of the general problem considered in section 3. In section 4 we consider the disturbances produced by the application of a constant step in strain on the boundary of the half-space. We find that the temperature is continuous and stress and strain are discontinuous at the wavefronts, the magnitudes of discontinuities of strain being different from those of stress. This is in contrast to the result obtained in the context of the Lord-Shulman theory (Norwood and Warren 1969) wherein the temperature, strain and stress are all discontinuous at the wavefronts and the magnitudes of discontinuities of strain are equal to those of stress. In section 5 we consider the disturbances produced by the application of a constant step in temperature on the boundary of the half-space. We find, as in Norwood-Warren (1969), that the temperature, strain and stress are all discontinuous at the wavefronts. However unlike in Norwood-Warren (1969), the magnitudes of discontinuities of stress differ from those of strain.

In section 6 we consider V. I. Danilvoskaya's prescribed traction problem. In Boley-Tolins (1962) and Norwood-Warren (1969) the solution of this problem was obtained by a superposition of solutions of prescribed strain and prescribed temperature problems. This procedure cannot be adopted in the present analysis because of the dependence of stresses on the temperature-velocity. We obtain the solution directly and find that the temperature, strain and stress are all discontinuous at the wavefronts. The dependence of stress on the temperature-velocity has a dominating influence on the magnitudes of discontinuities of stress and strain.

In order to compare our results with those obtained earlier, we consider in section 7 the long time approximation of the solution. We find that under this approximation, the relaxation times have no influence on the disturbances and that our results are identical with the corresponding results obtained in the context of the coupled theory (Boley and Tolins 1962). This supports the argument that the "second-sound" effects are short lived.

Several other results obtained in the course of our analysis are recorded at appropriate places.

2. Basic Equations

The equations governing linear thermoelastic interactions in a homogeneous and isotropic solid, as proposed by Green and Lindsay (1972) and Green (1972) are:

(a) the strain-displacement relations:

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \ldots(2.1)$$

(b) the stress-strain-temperature relations:

$$t_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} - (\beta/\kappa)(\theta + \alpha\dot{\theta})\,\delta_{ij} \qquad \ldots(2.2)$$

(c) the equation of motion:

$$v_2^2\,\nabla^2\mathbf{u} + (v_1^2 - v_2^2)\,\nabla\,\mathrm{div}\,\mathbf{u} - (\beta/\rho\kappa)\,\nabla(\theta + \alpha\dot{\theta}) = \ddot{\mathbf{u}} \qquad \ldots(2.3)$$

(d) the equation of heat conduction:

$$(k/\rho c)\,\nabla^2\theta - (\dot{\theta} + \alpha^*\ddot{\theta}) - (\beta\theta_0/\rho c\kappa)\,\dot{e}_{kk} = 0. \qquad \ldots(2.4)$$

The notation and symbols in these equations are as in Green (1972). Introducing the quantities

$$x_i' = \frac{\rho c v_1}{k} x_i, \quad t' = \frac{\rho c v_1^2}{k} t, \quad u_i' = \frac{\rho c v_1}{k} u_i$$

$$\theta' = \frac{\beta\theta}{\rho\kappa v_1^2}, \quad e_{ij}' = e_{ij}, \quad t_{ij}' = \frac{t_{ij}}{\rho v_1^2},$$

$$v_2' = v_2/v_1$$

into eqns. (2.1) – (2.4) and supressing primes we obtain the following equations which are in dimensionless form:

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad ...(2.5)$$

$$t_{ij} = (1 - 2v_2^2)\, e_{kk}\, \delta_{ij} + 2v_2^2 e_{ij} - (\theta + \delta\dot{\theta})\, \delta_{ij} \qquad ...(2.6)$$

$$v_2^2 \nabla^2 \mathbf{u} + (1 - v_2^2) \nabla \operatorname{div} \mathbf{u} - \nabla(\theta + \delta\dot{\theta}) = \ddot{\mathbf{u}} \qquad ...(2.7)$$

and

$$\nabla^2\theta - (\dot{\theta} + \eta\ddot{\theta}) - \epsilon\dot{e}_{kk} = 0 \qquad ...(2.8)$$

where we have put

$$\epsilon = \frac{\beta^2\theta_0}{\rho^2 c\kappa^2 v_1^2}, \; \delta = \frac{\rho c v_1^2}{k}\alpha, \; \eta = \frac{\rho c v_1^2}{k}\alpha^*. \qquad ...(2.9)$$

Clearly ϵ is the usual thermoelastic coupling factor. Since ρ, c, k, v_1^2 are all positive and $\alpha \geqslant \alpha^* \geqslant 0$ (Green 1972), it follows that δ and η satisfy the inequalities

$$\delta \geqslant \eta \geqslant 0. \qquad ...(2.10)$$

From eqns. (2.6) and (2.8) we see that if $\delta \neq 0$, the stresses depend on the temperature-velocity and if $\eta \neq 0$, the heat propagates with a finite speed. Since $\eta \neq 0$ implies $\delta \neq 0$, because of (2.10), it follows that the heat cannot propagate with a finite speed, unless the stresses depend on the temperature-velocity.

In the limiting case when $\delta \to 0$, eqns. (2.5) – (2.8) reduce to the governing equations of the coupled theory.

3. One-dimensional Deformation of a Half-space

We now consider one-dimensional deformation in the $z(= x_3)$-direction of the half-space $z \geqslant 0$ under the following assumptions:

(i) At time $t = 0$, the half-space is at rest in its undeformed state and it is maintained at a uniform temperature and zero temperature-velocity.

(ii) At time $t > 0$, the boundary $z = 0$ of the half-space experiences a uniformly distributed time-dependent strain or temperature change.

Then we have $\mathbf{u} = (0, 0, u(z, t))$ and $\theta = \theta(z, t)$. Equations (2.5) – (2.8) now yield the following governing equations of the problem:

$$e_{ij} = \begin{cases} e = \dfrac{\partial u}{\partial z}, \; i = j = 3 \\ 0, \; \text{otherwise} \end{cases} \qquad \text{...(3.1)}$$

$$t_{ij} = \begin{cases} \tau = e - (\theta + \delta\dot{\theta}), \; i = j = 3 \\ 0, \; \text{otherwise} \end{cases} \qquad \text{...(3.2)}$$

$$\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial z}(\theta + \delta\dot{\theta}) = \ddot{u} \qquad \text{...(3.3)}$$

$$\frac{\partial^2 \theta}{\partial z^2} - (\dot{\theta} + \eta\ddot{\theta}) - \epsilon\dot{e} = 0. \qquad \text{...(3.4)}$$

The relevant initial and boundary conditions are

$$\theta(z, 0) = \dot{\theta}(z, 0) = u(z, 0) = \dot{u}(z, 0) = 0 \qquad \text{...(3.5)}$$

$$e(0, t) = E(t), \quad \theta(0, t) = \Theta(t) \qquad \text{...(3.6)}$$

where $E(t)$ and $\Theta(t)$ are known functions.

We now solve the eqns. (3.1) – (3.4) under the conditions (3.5) and (3.6) by the use of the Laplace transform on t and the sine and cosine transforms on z. Applying the Laplace transform to eqns. (3.1) – (3.4), subject to the initial conditions (3.5), and then taking the sine transform of (3.1) and (3.4) and the cosine transform of (3.3), subject to the boundary conditions (3.6), and finally inverting the sine transforms (after eliminating the cosine transform), as explained by Norwood and Warren (1969), we obtain the following transform solutions for temperature, strain and stress:

$$\begin{aligned} \hat{\theta}(z, p) = {} & \frac{\epsilon p \hat{E}(p)}{r_1^2 - r_2^2} [\exp(-r_1 z) - \exp(-r_2 z)] + \frac{\hat{\Theta}(p)}{r_1^2 - r_2^2} \\ & \times [\{\epsilon p(1 + \delta p) + p^2 - r_2^2\} \exp(-r_2 z) \\ & \quad - \{\epsilon p(1 + \delta p) + p^2 - r_1^2\} \exp(-r_1 z)] \end{aligned} \qquad \text{...(3.7)}$$

$$\begin{aligned} \hat{e}(z, p) = {} & \frac{\hat{E}(p)}{r_1^2 - r_2^2} [\{p(1 + \eta p) - r_2^2\} \exp(-r_2 z) - \{p(1 + \eta p) - r_1^2\} \\ & \times \exp(-r_1 z)] + \frac{p(1 + \delta p)(1 + \eta p)}{r_1^2 - r_2^2} \hat{\Theta}(p) \\ & \times [\exp(-r_1 z) - \exp(-r_2 z)] \end{aligned} \qquad \text{(3.8)}$$

$$\hat{\tau}(z, p) = \hat{e}(z, p) - (1 + \delta p)\,\hat{\theta}(z, p). \qquad \text{...(3.9)}$$

Here $\hat{f}(z, p)$ is the Laplace transform of $f(z, t)$ and r_1, r_2 satisfy the relations

$$\left.\begin{aligned} r_1^2 + r_2^2 &= p\,[1 + \epsilon + p(1 + \eta + \epsilon\delta)] \\ r_1^2 - r_2^2 &= p\,[\{1 + \epsilon + p(1 + \eta + \epsilon\delta)\}^2 - 4p(1 + \eta p)]^{1/2}. \end{aligned}\right\} \qquad \text{...(3.10)}$$

The solutions (3.7) - (3.9) are not readily invertible in the variable p. Since the "second-sound" effects are small and short lived (Green and Lindsay 1972), we restrict ourselves mainly to the analysis of solutions (3.7) - (3.9) for small values of t. This corresponds to the case $p \to \infty$, as explained by Norwood and Warren (1969).

In the limit $p \to \infty$, we obtain from eqns. (3.10), the expressions

$$r_{1,2} = \frac{p}{V_{1,2}} + q_{1,2} + O\left(\frac{1}{p}\right) \qquad \text{...(3.11)}$$

$$(r_1^2 - r_2^2)^{-1} = p^{-3}\Delta^{-3/2}\left[p\Delta - L + O\left(\frac{1}{p}\right)\right] \qquad \text{...(3.12)}$$

where

$$\left.\begin{aligned} V_{1,2} &= \sqrt{2}\,[(1 + \eta + \epsilon\delta) \pm \Delta^{1/2}]^{-1/2} \\ q_{1,2} &= \tfrac{1}{4} V_{1,2}\,[1 + \epsilon \pm L\Delta^{-1/2}] \\ \Delta &= (1 - \eta + \epsilon\delta)^2 + 4\epsilon\delta\eta \\ L &= (1 + \epsilon)(\eta + \epsilon\delta) + \epsilon - 1. \end{aligned}\right\} \qquad \text{...(3.13)}$$

and

The form of eqn. (3.11) shows that the solutions (3.7) - (3.9) consist of two waves which propagate with speeds V_1 and V_2 given by (3.13_1). Noting that $V_1 \to 1$ and $V_2 \to \infty$ as $\delta \to 0$, we may infer that the wave which propagates with speed V_1 is acoustic and that which propagates with speed V_2 is thermal in nature. Further, since $V_1 < V_2$, the thermal wave precedes the acoustic wave. A consequence of this is that the points of the half-space for which $z > tV_2$ do not experience any disturbance. These results are similar to those obtained in the context of the Lord-Shulman theory (Lord and Shulman 1967, Norwood and Warren 1969).

We now consider a few particular cases.

Case (i): $\eta = 0$ — The expressions (3.13) now yield

$$V_1 = (1 + \epsilon\delta)^{-1/2},\; V_2 \to \infty. \qquad \text{...(3.14)}$$

Hence in this case only the acoustic wave propagates with a finite speed and it is slower than its counterpart in the coupled theory.

Case (*ii*) : $\delta = \eta \neq 0$ — The expressions (3.13) now yield $V_{1,2} = V^*_{1,2}$ which are given by the relations

$$\left. \begin{aligned} V^*_{1,2} &= \sqrt{2}\,[1 + \eta(1 + \epsilon) \pm \Gamma^{1/2}]^{-1/2} \\ \text{and} \qquad \Gamma &= (1 - \eta + \epsilon\eta)^2 + 4\epsilon\eta^2. \end{aligned} \right\} \qquad \ldots(3.15)$$

Identifying our relaxation constant η with the relaxation constant β of the Lord-Shulman theory we readily see that the expressions (3.15) are identical with the expressions obtained by Norwood and Warren (1969) for the speeds of propagation. Comparing the expressions (3.13) and (3.15) and noting that $\delta \geqslant \eta$, we find that $V_1 \leqslant V_1^*$. Accordingly, the acoustic wave of the present analysis propagates with a lesser speed than its counterpart in the Lord-Shulman theory, in general.

Case (*iii*) : $\epsilon = 0$ — The expressions (3.13) now yield

$$V_1 = 1,\; V_2 = \eta^{-1/2}. \qquad \ldots(3.16)$$

Lord and Shulman (1967) obtain these speeds by neglecting the dependence of energy on the strain-acceleration in their theory.

4. Step in Strain on the Boundary

Suppose that the disturbances in the half-space are due to the application of a constant step e_0 in strain on the boundary $z = 0$. Then we have

$$E(t) = e_0 H(t),\; \hat{E}(p) = \frac{e_0}{p},\; \Theta(t) = \hat{\Theta}(p) = 0$$

where $H(t)$ is the Heaviside unit function.

The solutions (3.7) – (3.9) now reduce to

$$\hat{\theta}(z, p) = \frac{\epsilon e_0}{r_1^2 - r_2^2}\,[\exp(-r_1 z) - \exp(-r_2 z)] \qquad \ldots(4.1)$$

$$\hat{e}(z, p) = \frac{e_0}{p(r_1^2 - r_2^2)}\,[\{p(1 + \eta p) - r_2^2\}\exp(-r_2 z) - \{p(1 + \eta p) - r_1^2\}\exp(-r_1 z)] \qquad \ldots(4.2)$$

$$\hat{\tau}(z, p) = \frac{e_0}{p(r_1^2 - r_2^2)}\,[\{p(1 + \eta p) + \epsilon p(1 + \delta p) - r_2^2\}\exp(-r_2 z) - \{p(1 + \eta p) + \epsilon p(1 + \delta p) - r_1^2\}\exp(-r_1 z)]. \qquad \ldots(4.3)$$

In the short time approximation (i.e. when $p \to \infty$), the inversions of the solutions (4.1) – (4.3) may be obtained, on using (3.11) – (3.13), in the following form:

$$\theta(z, t) = e_0\epsilon\Delta^{-1/2}\left[\left(t - \frac{z}{V_1}\right) H\left(t - \frac{z}{V_1}\right) \exp(-q_1 z) - \left(t - \frac{z}{V_2}\right) H\left(t - \frac{z}{V_2}\right) \exp(-q_2 z)\right] \quad \ldots(4.4)$$

$$e(z, t) = \tfrac{1}{2} e_0 \Delta^{-1/2}\left[\left\{\Delta^{1/2} + (1 - \eta + \epsilon\delta) - 2\epsilon M \Delta^{-1}\left(t - \frac{z}{V_1}\right)\right\} \times H\left(t - \frac{z}{V_1}\right) \exp(-q_1 z) + \left\{\Delta^{1/2} - (1 - \eta + \epsilon\delta) + 2\epsilon M \Delta^{-1}\left(t - \frac{z}{V_2}\right)\right\} H\left(t - \frac{z}{V_2}\right) \exp(-q_2 z)\right] \quad \ldots(4.5)$$

$$\tau(z, t) = \tfrac{1}{2} e_0 \Delta^{-1/2}\left[\left\{\Delta^{1/2} + (1 - \eta - \epsilon\delta) - 2\epsilon N \Delta^{-1}\left(t - \frac{z}{V_1}\right)\right\} \times H\left(t - \frac{z}{V_1}\right) \exp(-q_1 z) + \left\{\Delta^{1/2} - (1 - \eta - \epsilon\delta) + 2\epsilon N \Delta^{-1}\left(t - \frac{z}{V_2}\right)\right\} H\left(t - \frac{z}{V_2}\right) \exp(-q_2 z)\right] \quad \ldots(4.6)$$

where

$$\left.\begin{aligned} M &= \delta + \eta + (\delta - \eta)(\eta + \epsilon\delta) \\ N &= 1 - \eta + \delta(2 + \epsilon). \end{aligned}\right\} \quad \ldots(4.7)$$

In the solutions (4.4) – (4.6), the first term represents the contribution of the acoustic wave in the vicinity of its wavefront, viz., $z = V_1 t$ and the second term represents the contribution of the thermal wave in the vicinity of its wavefront, viz., $z = V_2 t$.

The eqn. (4.4) shows that the temperature is continuous at both the wavefronts. This is in contrast to the result obtained by Norwood and Warren (1969) where a temperature discontinuity exists at each of the wavefronts. This is because of the fact that the Lord-Shulman theory takes account of the effect of strain-acceleration on the thermal field whereas the Green-Lindsay theory does not consider any such effect.

Equations (4.5) and (4.6) show that both the strain and stress experience discontinuities at each of the wavefronts, as in Norwood and Warren (1969). The magnitudes of these discontinuities are given by

$$[e^+ - e^-]_{z=tV_{1,2}} = \tfrac{1}{2} e_0 \{1 \pm \Delta^{-1/2}(1 - \eta + \epsilon\delta)\} \exp(-q_{1,2} z) \quad \ldots(4.8)$$

$$[\tau^+ - \tau^-]_{z=tV_{1,2}} = \tfrac{1}{2} e_0 \{1 \pm \Delta^{-1/2}(1 - \eta - \epsilon\delta)\} \exp(-q_{1,2} z) \quad \ldots(4.9)$$

In view of the Tauberian theorem for hyperbolic equations (Knopoff and Gilbert 1959) we note, as in Norwood and Warren (1969), that although the expressions

(4.4) – (4.6) are just the short time approximations, the magnitudes of discontinuities given by (4.8) and (4.9) are exact. We readily see that the magnitudes of discontinuities in strain differ from those in stress, in general. This is in contrast to the result in the Lord-Shulman theory wherein the magnitudes of discontinuities in strain are equal to those in stress. Obviously, this is due to the dependence of stresses on the temperature-velocity.

By neglecting the coupling between the thermal field and the strain-acceleration in their theory, Lord and Shulman (1967) show that the temperature is continuous at both the wavefronts (as in the present analysis) and that the strain is continuous at the thermal wavefront and discontinuous at the acoustic wavefront. Hence, the discontinuities in strain and stress which occur at the thermal wavefront in our analysis are also due to the dependence of stress on the temperature-velocity.

We consider a few particular cases.

Case (i) : $\eta = 0$ — In this case we have $V_1 = (1 + \epsilon\delta)^{-1/2}$ and $V_2 \to \infty$, vide eqns. (3.14). The expressions (4.8) and (4.9) now reduce with the aid of (3.13) to

$$[e^+ - e^-]_{z=tV_1} = e_0 \exp\left\{-\frac{\epsilon}{2}\left(1 + \frac{\delta}{1 + \epsilon\delta}\right) V_1 z\right\} \quad ...(4.10)$$

$$[\tau^+ - \tau^-]_{z=tV_1} = \frac{1}{1 + \epsilon\delta}[e^+ - e^-]_{z=tV_1}. \quad ...(4.11)$$

In the limiting case when $\delta \to 0$, these reduce to

$$[e^+ - e^-]_{z=t} = [\tau^+ - \tau^-]_{z=t} = e_0 \exp\left(-\frac{\epsilon}{2} z\right)$$

which are in agreement with the expressions obtained in the context of the coupled theory (Boley and Tolins 1962).

Case (ii) : $\delta = \eta \neq 0$ — In this case eqns. (4.5), (4.6), (4.8) and (4.9) reduce to those obtained in the context of the Lord-Shulman theory (Norwood and Warren 1969).

Case (iii) : $\epsilon = 0$ — In this case we have $V_1 = 1$ and $V_2 = \eta^{-1/2}$ vide eqns. (3.16). Expressions (3.13), (4.8) and (4.9) now show that the strain and stress are continuous at the thermal wavefront and that these experience discontinuities of magnitude e_0 at the acoustic wavefront.

5. Step in Temperature on the Boundary

Suppose that the disturbances in the half-space are due to the application of a constant step T_0 in the temperature on the boundary. Then we have

$$E(t) = \hat{E}(p) = 0,\ \Theta(t) = T_0 H(t),\ \hat{\Theta}(p) = \frac{T_0}{p}.$$

The solutions (3.7) – (3.9) now reduce to

$$\hat{\theta}(z, p) = \frac{T_0}{p(r_1^2 - r_2^2)} [\{r_1^2 - \epsilon p(1 + \delta p) - p^2\} \exp(-r_1 z) - \{r_2^2 - \epsilon p(1 + \delta p) - p^2\} \exp(-r_2 z)] \quad ...(5.1)$$

$$\hat{e}(z, p) = \frac{T_0(1 + \delta p)(1 + \eta p)}{r_1^2 - r_2^2} [\exp(-r_1 z) - \exp(-r_2 z)] \quad ...(5.2)$$

$$\hat{\tau}(z, p) = \frac{T_0(1 + \delta p)}{p(r_1^2 - r_2^2)} [\{p(1 + \eta p) + \epsilon p(1 + \delta p) + p^2 - r_1^2\} \exp(-r_1 z) - \{p(1 + \eta p) + \epsilon p(1 + \delta p) + p^2 - r_2^2\} \exp(-r_2 z)]. \quad ...(5.3)$$

In the short time approximation the inversions of these expressions are given by [on using (3.11) – (3.13)],

$$\theta(z, t) = \tfrac{1}{2} T_0 \Delta^{-1/2} \left[\{\Delta^{1/2} - (1 - \eta + \epsilon\delta)\} + 2\epsilon M \Delta^{-1}\left(t - \frac{z}{V_1}\right)\right] \times H\left(t - \frac{z}{V_1}\right) \exp(-q_1 z) + \tfrac{1}{2} T_0 \Delta^{-1/2} \left[\{\Delta^{1/2} + (1 - \eta + \epsilon\delta)\} - 2\epsilon M \Delta^{-1}\left(t - \frac{z}{V_2}\right)\right] H\left(t - \frac{z}{V_2}\right) \exp(-q_2 z) \quad ...(5.4)$$

$$e(z, t) = T_0 \Delta^{-1/2} \left[\delta + \eta - \delta\eta L \Delta^{-1} + \{1 - (\delta + \eta) L \Delta^{-1}\}\left(t - \frac{z}{V_1}\right)\right] \times H\left(t - \frac{z}{V_1}\right) \exp(-q_1 z) - T_0 \Delta^{-1/2} \left[\delta + \eta - \delta\eta L \Delta^{-1} + \{1 - (\delta + \eta) L \Delta^{-1}\}\left(t - \frac{z}{V_2}\right)\right] H\left(t - \frac{z}{V_2}\right) \exp(-q_2 z) \quad ...(5.5)$$

$$\tau(z, t) = \tfrac{1}{2} T_0 \Delta^{-1/2} \Big[\delta(1 + \epsilon - L \Delta^{-1/2}) + (1 - \delta L \Delta^{-1}) \times (1 + \eta + \epsilon\delta - \Delta^{1/2}) \{1 + \epsilon + \tfrac{1}{2}\delta(1 + \epsilon)^2 \Delta^{-1/2} - L \Delta^{-1}(1 + \delta + \eta + 2\epsilon\delta - \delta L \Delta^{-1/2})\} \times \left(t - \frac{z}{V_1}\right)\Big] H\left(t - \frac{z}{V_1}\right) \exp(-q_1 z) -$$

(*equation continued on p.* 236)

$$- \tfrac{1}{2} T_0 \Delta^{-1/2} \Big[\delta(1 + \epsilon + L\Delta^{-1/2}) + (1 - \delta L \Delta^{-1})$$
$$\times (1 + \eta + \epsilon\delta + \Delta^{1/2}) + \{1 + \epsilon$$
$$- \tfrac{1}{2}\delta(1 + \epsilon)^2 \Delta^{-1/2} + L\Delta^{-1}(1 + \eta - \delta - \delta L \Delta^{-1/2})\}$$
$$\times \left(t - \frac{z}{V_2}\right)\Big] H\left(t - \frac{z}{V_2}\right) \exp(-q_2 z). \qquad ...(5.6)$$

These equations show that, as in the Lord-Shulman theory (Norwood and Warren 1969), the temperature, strain and stress experience discontinuities at each of the wavefronts. The magnitudes of these discontinuities are given by

$$[\theta^+ - \theta^-]_{z=tV_{1,2}} = \tfrac{1}{2} T_0 \,[1 \mp \Delta^{-1/2}(1 - \eta + \epsilon\delta)] \exp(-q_{1,2} z) \qquad ...(5.7)$$

$$[e^+ - e^-]_{z=tV_{1,2}} = \pm\, T_0 \Delta^{-1/2}(\delta + \eta - \delta\eta L \Delta^{-1}) \exp(-q_{1,2} z) \qquad ...(5.8)$$

$$[\tau^+ - \tau^-]_{z=tV_{1,2}} = \pm \tfrac{1}{2} T_0 \Delta^{-1/2} \,[\delta(1 + \epsilon \mp \Delta^{-1/2} L)$$
$$+ (1 - \delta L \Delta^{-1})(1 + \eta + \epsilon\delta \mp \Delta^{1/2})] \exp(-q_{1,2} z). \qquad ...(5.9)$$

We readily see that the magnitudes of discontinuities in strain are in general different from those in stress, unlike in the Lord-Shulman theory.

We consider the following particular cases.

Case (*i*) : $\eta = 0$ — In this case we have $V_1 = (1 + \epsilon\delta)^{-1/2}$ and $V_2 \to \infty$. We now verify from the expressions (3.13) and (5.7) - (5.9) that the temperature is continuous at the acoustic wavefront and that the magnitudes of discontinuities in strain and stress at this wavefront are given by

$$[e^+ - e^-]_{z=tV_1} = \frac{T_0\delta}{1 + \epsilon\delta} \exp\left\{-\frac{\epsilon}{2}\left(1 + \frac{\delta}{1 + \epsilon\delta}\right) V_1 z\right\}$$

$$[\tau^+ - \tau^-]_{z=tV_1} = \frac{1}{1 + \epsilon\delta} \cdot [e^+ - e^-]_{z=tV_1}.$$

Clearly these discontinuities are due to dependence of stress on the temperature-velocity. In the limiting case when $\delta \to 0$, the strain and stress are also continuous at the acoustic wavefront. This agrees with the corresponding result in the coupled theory (Boley and Tolins 1962).

Case (*ii*) : $\delta = \eta \neq 0$ — In this case the expressions (5.1), (5.4) and (5.7) which are associated with the thermal field are identical with those obtained in the context of the Lord-Shulman theory (Norwood and Warren 1969). The corresponding expressions for strain (and therefore stress) differ substantially from those obtained by Norwood and Warren (1969).

Case (iii) : $\epsilon = 0$ — In this case we have $V_1 = 1$ and $V_2 = \eta^{-1/2}$. We verify from eqns. (5.7) – (5.9) along with (3.13) that the temperature is continuous at the acoustic wavefront and that the magnitudes of other discontinuities are given by

$$[\theta^+ - \theta^-]_{z=tV_2} = T_0 \exp\left(-\frac{z}{2\sqrt{\eta}}\right)$$

$$[e^+ - e^-]_{z=tV_1} = [\tau^+ - \tau^-]_{z=tV_1} = \frac{T_0}{1-\eta}\left(\eta + \frac{\delta}{1-\eta}\right)$$

$$[e^+ - e^-]_{z=tV_2} = [\tau^+ - \tau^-]_{z=tV_2} = \frac{T_0}{\eta - 1}\left(\eta + \frac{\delta}{1-\eta}\right)\exp\left(-\frac{z}{2\sqrt{\eta}}\right).$$

Clearly these discontinuities disappear as $\delta \to 0$.

6. Zero Stress on the Boundary (Danilovskaya's Problem)

Suppose that a constant step T_0 in temperature is applied to the boundary which is maintained stress-free. Then we have $\tau(0, t) = 0$, $\theta(0, t) = T_0 H(t)$. These together with eqn. (3.9) yields

$$\hat{e}(0, p) = (1 + \delta p)\,\hat{\theta}(0, p), \quad \hat{\theta}(0, p) = \frac{T_0}{p}$$

Hence we have in this case

$$\hat{E}(p) = \frac{1}{p} T_0 (1 + \delta p), \quad \hat{\Theta}(p) = \frac{T_0}{p}$$

The solutions (3.7) – (3.9) now reduce to

$$\hat{\theta}(z, p) = \frac{T_0}{r_1^2 - r_2^2}\left[\left(\frac{r_1^2}{p} - p\right)\exp(-r_1 z) - \left(\frac{r_2^2}{p} - p\right)\exp(-r_2 z)\right] \quad \text{...(6.1)}$$

$$\hat{e}(z, p) = \frac{T_0(1 + \delta p)}{p(r_1^2 - r_2^2)}\left[r_1^2 \exp(-r_1 z) - r_2^2 \exp(-r_2 z)\right] \quad \text{...(6.2)}$$

$$\hat{\tau}(z, p) = \frac{T_0 p(1 + \delta p)}{r_1^2 - r_2^2}\left[\exp(-r_1 z) - \exp(-r_2 z)\right] \quad \text{...(6.3)}$$

In the short time approximation the inversions of these may be obtained, on using (3.11) – (3.13), in the following form.

$$\theta(z, t) = \tfrac{1}{2} T_0 \Delta^{-1/2}\Bigg[\left\{\Delta^{1/2} - (1 - \eta - \epsilon\delta) + 2\epsilon N \Delta^{-1}\left(t - \frac{z}{V_1}\right)\right\} \times H\left(t - \frac{z}{V_1}\right)\exp(-q_1 z) + \Big\{\Delta^{1/2} + (1 - \eta - \epsilon\delta) - 2\epsilon N \Delta^{-1}\left(t - \frac{z}{V_2}\right)\Big\} H\left(t - \frac{z}{V_2}\right)\exp(-q_2 z)\Bigg] \quad \text{...(6.4)}$$

$$
\begin{aligned}
e(z,t) = \tfrac{1}{2} T_0 \Delta^{-1/2} \Big[& \delta(1+\epsilon+L\Delta^{-1/2}) + (1-\delta L\Delta^{-1}) \\
& \times (1+\eta+\epsilon\delta+\Delta^{1/2}) \\
& + \{\tfrac{1}{2}\delta\Delta^{-1/2}(1+\epsilon)^2 + (1-\delta L\Delta^{-1})(1+\epsilon+L\Delta^{-1/2}) \\
& - L\Delta^{-1}(1+\eta+\epsilon\delta+\Delta^{1/2})\}\left(t-\frac{z}{V_1}\right)\Big] H\left(t-\frac{z}{V_1}\right)\exp(-q_1 z) \\
& - \tfrac{1}{2} T_0 \Delta^{-1/2} \Big[\delta(1+\epsilon-L\Delta^{-1/2}) + (1-\delta L\Delta^{-1}) \\
& \times (1+\eta+\epsilon\delta-\Delta^{1/2}) \\
& - \{\tfrac{1}{2}\delta\Delta^{-1/2}(1+\epsilon)^2 - (1-\delta L\Delta^{-1})(1+\epsilon-L\Delta^{-1/2}) \\
& + L\Delta^{-1}(1+\eta+\epsilon\delta-\Delta^{1/2})\}\left(t-\frac{z}{V_2}\right)\Big] H\left(t-\frac{z}{V_2}\right)\exp(-q_2 z)
\end{aligned}
\quad \ldots(6.5)
$$

$$
\begin{aligned}
\tau(z,t) = T_0 \Delta^{-1/2} \Big[& \Big\{(1-\delta L\Delta^{-1}) - L\Delta^{-1}\left(t-\frac{z}{V_1}\right)\Big\} \\
& \times H\left(t-\frac{z}{V_1}\right)\exp(-q_1 z) - \Big\{(1-\delta L\Delta^{-1}) \\
& - L\Delta^{-1}\left(t-\frac{z}{V_2}\right)\Big\} H\left(t-\frac{z}{V_2}\right)\exp(-q_2 z)\Big].
\end{aligned}
\quad \ldots(6.6)
$$

These expressions show that the temperature, strain and stress experience discontinuities at both the wavefronts. The magnitudes of these discontinuities are given by

$$[\theta^+ - \theta^-]_{z=tV_{1,2}} = \tfrac{1}{2} T_0 \,[1 \mp \Delta^{-1/2}(1-\eta-\epsilon\delta)] \exp(-q_{1,2} z) \quad \ldots(6.7)$$

$$
\begin{aligned}
[e^+ - e^-]_{z=tV_{1,2}} = \tfrac{1}{2} T_0 \,[1 \pm \Delta^{-1/2} & \{\delta(1+\epsilon) \\
& + (1-\delta L\Delta^{-1})(1+\eta+\epsilon\delta)\}] \exp(-q_{1,2} z)
\end{aligned}
\quad \ldots(6.8)
$$

$$[\tau^+ - \tau^-]_{z=tV_{1,2}} = \pm\, T_0 \Delta^{-1/2} (1-\delta L\Delta^{-1}) \exp(-q_{1,2} z). \quad \ldots(6.9)$$

Clearly, in this case also the magnitudes of discontinuities in strain differ from those of stress, unlike in the Lord-Shulman theory.

We consider the following particular cases.

Case (*i*) : $\eta = 0$ — Expressions (6.7) - (6.9) now reduce with the aid of (3.13) to

$$[\theta^+ - \theta^-]_{z=tV_1} = \frac{T_0\epsilon\delta}{1+\epsilon\delta} \exp\left\{-\frac{\epsilon}{2}\left(1+\frac{\delta}{1+\epsilon\delta}\right) V_1 z\right\}.$$

$$[e^+ - e^-]_{z=tV_1} = T_0\left[1 + \frac{\delta}{(1+\epsilon\delta)^2}\right]\exp\left\{-\frac{\epsilon}{2}\left(1+\frac{\delta}{1+\epsilon\delta}\right)V_1 z\right\}$$

$$[\tau^+ - \tau^-]_{z=tV_1} = \frac{T_0\,[1+\delta(1+\epsilon-\epsilon\delta)]}{(1+\epsilon\delta)^3} \times \exp\left\{-\frac{\epsilon}{2}\left(1+\frac{\delta}{1+\epsilon\delta}\right)V_1 z\right\}$$

In the limiting case when $\delta \to 0$, the temperature is continuous at the acoustic wavefront, and

$$[e^+ - e^-]_{z=tV_1} = [\tau^+ - \tau^-]_{z=tV_1} = T_0 \exp\left(-\frac{\epsilon}{2}\,z\right)$$

These are in agreement with those obtained in the context of the coupled theory (Boley and Tolins 1962).

Case (ii) $\delta = \eta \neq 0$ — In this case the magnitudes of discontinuities in temperature given by (6.7) are identical with those obtained by Popov (1967) and Norwood and Warren (1969). However, the magnitudes of discontinuities in strain (and therefore stress) differ from those obtained there. Note that the dependence of stress on the temperature-velocity has a dominating influence on the discontinuities in stress and strain.

Case (iii) : $\epsilon = 0$ — In this case we verify that the temperature is continuous at the acoustic wavefront and that the magnitudes of other discontinuities are given by

$$[\theta^+ - \theta^-]_{z=tV_2} = T_0 \exp\left(-\frac{z}{2\sqrt{\eta}}\right)$$

$$[e^+ - e^-]_{z=tV_1} = [\tau^+ - \tau^-]_{z=tV_1} = \frac{T_0}{1-\eta}\left(1 + \frac{\delta}{1-\eta}\right)$$

$$[e^+ - e^-]_{z=tV_2} = \frac{T_0}{\eta - 1}\left[\delta + \eta\left(1 + \frac{\delta}{1-\eta}\right)\right]\exp\left(-\frac{z}{2\sqrt{\eta}}\right)$$

$$[\tau^+ - \tau^-]_{z=tV_2} = \frac{T_0}{\eta - 1}\left[1 + \frac{\delta}{1-\eta}\right]\exp\left(-\frac{z}{2\sqrt{\eta}}\right).$$

Note that if the heat propagates with infinite speed, only stress and strain experience a discontinuity (of magnitude T_0) at the acoustic wavefront.

7. Long Time Approximation

In this section we consider the long time approximation of the solutions (3.7) – (3.9) and their particular cases considered in sections 4–6. As is well known in the Laplace transform theory, this approximation corresponds to $p \to 0$.

In the limit $p \to 0$, the expressions (3.10) yield

$$r_1 = p^{1/2}(1 + \epsilon)^{1/2}, \quad r_2 = 0. \qquad \text{...(7.1)}$$

In the case of a step in strain on the boundary, the solutions (4.1) – (4.3) yield the following expressions.

$$\left.\begin{aligned} \theta(z, t) &= -\frac{e_0\epsilon}{1+\epsilon}\operatorname{erf}\left\{\frac{z(1+\epsilon)^{1/2}}{2t^{1/2}}\right\} \\ e(z, t) &= e_0\left[1 - \frac{\epsilon}{1+\epsilon}\operatorname{erf}\left\{\frac{z(1+\epsilon)^{1/2}}{2t^{1/2}}\right\}\right] \\ \tau(z, t) &= e_0 \end{aligned}\right\} \qquad \text{...(7.2)}$$

where erf x is the error function.

In the case of a step in temperature on the boundary, the solutions (5.1) – (5.3) along with (7.1) yield the following expressions.

$$\left.\begin{aligned} e(z, t) &= -\frac{T_0}{1+\epsilon}\operatorname{erf}\left\{\frac{z(1+\epsilon)^{1/2}}{2t^{1/2}}\right\} \\ \theta(z, t) &= T_0\left[1 - \frac{1}{1+\epsilon}\operatorname{erf}\left\{\frac{z(1+\epsilon)^{1/2}}{2t^{1/2}}\right\}\right] \\ \tau(z, t) &= -T_0. \end{aligned}\right\} \qquad \text{...(7.3)}$$

In the case of Danilvoskaya's problem, the solutions (6.1) – (6.3) along with (7.1) yield the following expressions :

$$\left.\begin{aligned} \theta(z, t) = e(z, t) &= T_0\left[1 - \operatorname{erf}\left\{\frac{z(1+\epsilon)^{1/2}}{2t^{1/2}}\right\}\right] \\ \tau(z, t) &= 0. \end{aligned}\right\} \qquad \text{...(7.4)}$$

The expressions (7.2) – (7.4) are identical with those obtained in the context of the coupled theory (Boley and Tolins 1967). In view of this we may conclude that under long time approximation the relaxation times have no influence on the disturbances. This supports the argument that 'second-sound' effects are short lived.

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