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DIFFRACTION OF SH WAVES BY TWO COPLANAR GRIFFITH CRACKS AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC HALF-SPACES

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The problem of diffraction of normally incident anti-plane shear waves by two parallel and coplanar Griffith cracks located at the interface of two bonded elastic solids of different elastic properties is solved. Approximate solutions valid for small wave frequencies have been obtained. The solutions are used to calculate the quantities of physical interest like stress-intensity factors at the inner and outer edges of the cracks.

1. INTRODUCTION

During recent years the problem of interaction of elastic waves with singularities in the form of cracks or inclusions located in various two or three dimensional configurations has attracted considerable interest. Although the diffraction of elastic waves by cracks located in homogeneous, isotropic elastic solids has been the subject of many investigations, to our knowledge the diffraction of elastic waves with cracks located at the interface of two bonded elastic half-spaces has not been investigated. First attempt in this direction was made by Srivastava *et al.* (1978, 1979). They studied the interaction of longitudinal P-waves by a Griffith or a penny-shaped crack situated at the interface of two dissimilar elastic half-spaces bonded together. This has been supplimented (see Srivastava *et al.* 1980a, b) by studying the interaction of antiplane shear waves (SH waves) by a Griffith crack and torsional waves by a penny-shaped crack located at the interface of dissimilar elastic half-spaces.

In continuation to the above, our object here is to study the interaction of antiplane shear waves by two coplanar Griffith cracks situated at the interface of two bonded dissimilar elastic solids. In our previous papers (Srivastava *et al.* 1980a, b), we obtained both iterative and numerical solutions of the governing integral equations. On comparing these solutions we derived the conclusion that the iterative solution of the integral equations are valid for the values of dimensionless wave frequency $k_1 < 0.6$. For majority of elastic materials, the shear wave velocity is greater than 2×10^5 cm/sec. If the crack length is assumed to be of the order of 10^{-4} cm, the circular frequency can be as high as 10^9 cycles/sec for $k_1 < 0.6$. Hence for most practical cases the approximate iterative solutions are sufficiently accurate to predict the behaviour of the crack under dynamic loading.

In view of the above, in this paper we have obtained only the iterative solution of the integral equation and used it to calculate analytical expressions for stress intensity factors at the edge of the crack.

2. Formulation of the Problem

Let two parallel Griffith cracks of infinite length and finite width be located at the interface of two bonded dissimilar elastic semi-infinite solids. Consider a rectangular Cartesian coordinate system (x_1, x_2, x_3) such that these cracks occupy the region $-a \leqslant x_1 \leqslant -b, b \leqslant x_1 \leqslant a, -\infty < x_2 < \infty, x_3 = 0$ at the interface of two half-spaces $x_3 \geqslant 0$ and $x_3 \leqslant 0$. The crack is assumed to be excited by a normally incident antiplane shear wave originating at $x_3 = -\infty$. The displacement vector corresponding to this wave is parallel to x_2-axis. It is convenient to normalize all lengths with respect to 'a' which is half of the distance between the outer edges of these cracks. Writing $x_1/a = x$, $x_2/a = y$, $x_3/a = z$ and $b/a = c$ the cracks are defined by $-1 \leqslant x \leqslant -c, c \leqslant x \leqslant 1, -\infty < y < \infty, z = 0$.

Thus we have the problem of finding stress distribution when the cracks are subjected to the following boundary conditions :

$$\sigma_{yz}(x, 0+) = \sigma_{yz}(x, 0-) = -p_s - p_0 \exp(-i\omega t), \quad c \leqslant |x| \leqslant 1 \quad \ldots(2.1)$$

$$\sigma_{yz}(x, 0+) = \sigma_{yz}(x, 0-), \quad |x| < c, \quad |x| > 1 \quad \ldots(2.2)$$

$$U_y(x, 0+) = U_y(x, 0-), \quad |x| < c, \quad |x| > 1 \quad \ldots(2.3)$$

where ω is the circular frequency and p_s is the static pressure which is assumed to be sufficiently large so that crack faces do not come in contact during vibration. Since the solution of elastic problems may be superimposed, the static pressure p_s may be dropped. In the later part of the discussion the time factor $\exp(-i\omega t)$ will also be omitted but understood. The problem of finding stress distribution reduces to that of obtaining the solution of the displacement equation

$$U_{y,xx} + U_{y,zz} + k^2 U_y = 0 \quad \ldots(2.4)$$

where $k^2 = \rho\omega^2 a^2/\mu$, μ is Lamé s constant and ρ is the material density. The solution of this equation is

$$U_y(x, z) = \begin{cases} 2\int_0^\infty A_1(\xi) \exp(-\beta_1 z) \cos \xi x \, d\xi, & z > 0 \\ 2\int_0^\infty A_2(\xi) \exp(\beta_2 z) \cos \xi x \, d\xi, & z < 0 \end{cases} \quad \ldots(2.5)$$

where

$$\beta_j = \begin{cases} (\xi^2 - k_j^2)^{1/2}, & k_j < \xi \\ -i(k_j^2 - \xi^2)^{1/2}, & k_j > \xi \end{cases} \quad \ldots(2.6)$$

$$k_j^2 = \rho_j \omega^2 a^2/\mu_j, \quad j = 1, 2.$$

The suffixes 1 and 2 correspond to the half-spaces $z > 0$ and $z < 0$ respectively. In eqn. (2.5), $A_1(\xi)$ and $A_2(\xi)$ are unkown functions which shall be determined using the boundary conditions (2.1) – (2.3).

Corresponding to (2.5), following are expressions for stress components :

$$\sigma_{yz}(x, z) = \begin{cases} -2\mu_1 \int_0^\infty \beta_1 A_1(\xi) \exp(-\beta_1 z) \cos \xi x \, d\xi, & z > 0 \\ 2\mu_2 \int_0^\infty \beta_2 A_2(\xi) \exp(\beta_2 z) \cos \xi x \, d\xi, & z < 0. \end{cases} \quad \text{...(2.7)}$$

3. Derivation of Integral Equation

From the boundary conditions (2.1) and (2.2) we see that

$$\sigma_{yz}(x, 0+) = \sigma_{yz}(x, 0-)$$

for all values of x, hence we have

$$A_2(\xi) = -\frac{\mu_1\beta_1}{\mu_2\beta_2} A_1(\xi). \quad \text{...(3.1)}$$

The boundary conditions (2.1) and (2.3) lead to the following dual integral equations :

$$\int_0^\infty \beta_1 A_1(\xi) \cos \xi x \, d\xi = p_0/2\mu_1, \quad c < | x | < 1 \quad \text{...(3.2a)}$$

$$\int_0^\infty \beta_2^{-1} (\mu_1\beta_1 + \mu_2\beta_2) A_1(\xi) \cos \xi x \, d\xi = 0, \quad | x | < c, \quad | x | > 1 \quad \text{...(3.2b)}$$

Making the substitution

$$\beta_2^{-1} (\mu_1\beta_1 + \mu_2\beta_2) A_1(\xi) = B(\xi) \quad \text{...(3.3)}$$

the above integral equations can be written in a convenient form

$$\int_0^\infty \xi [1 + H(\xi)] B(\xi) \cos \xi x \, d\xi = (\mu_1 + \mu_2) p_0/2\mu_1, \quad c < | x | < 1. \quad \text{...(3.4)}$$

$$\int_0^\infty B(\xi) \cos \xi x \, d\xi = 0, \quad | x | < c, \quad | x | > 1 \quad \text{...(3.5)}$$

where

$$H(\xi) = (\mu_1 + \mu_2) \{(\mu_1\beta_1 + \mu_2\beta_2) \xi\}^{-1} \beta_1\beta_2 - 1. \quad \text{...(3.6)}$$

It can be easily demonstrated that $H(\xi) \to 0$ as $\xi \to \infty$. In order to solve the triple integral equation we set

$$B(\xi) = \xi^{-1} \int_c^1 h(t^2) \sin \xi t \, dt \qquad \ldots(3.7)$$

where the function $h(t^2)$ shall soon be determined. For the interval $|x| > 1$, eqn. (3.5) is automatically satisfied in view of the formula

$$\int_0^\infty \xi^{-1} \sin \xi t \cos \xi x \, d\xi = \begin{cases} \pi/2, & |x| < t \\ 0, & |x| > t \end{cases} \qquad \ldots(3.8)$$

and for the interval $0 \leqslant |x| < c$, eqn. (3.5) is satisfied provided

$$\int_c^1 h(t^2) \, dt = 0. \qquad \ldots(3.9)$$

On using the relations

$$\frac{\sin \xi x \sin \xi t}{\xi} = \frac{\pi}{2} (xt)^{1/2} J_{1/2}(\xi t) J_{1/2}(\xi x) \qquad \ldots(3.10)$$

and

$$J_{1/2}(\xi x) = \left(\frac{2\xi}{\pi x}\right)^{1/2} \int_0^x \frac{w J_0(\xi w)}{(x^2 - w^2)^{1/2}} dw \qquad \ldots(3.11)$$

eqn. (3.4) reduces to the following integro-differential equation for the determination of $h(t^2)$:

$$\int_c^1 \frac{t h(t^2) \, dt}{t^2 - x^2} = \frac{p_0(\mu_1 + \mu_2)}{2\mu_1} - \frac{d}{dx} \int_c^1 h(t^2)$$

$$\times \int_0^t \int_0^x \frac{y w L(y, w) \, dy \, dw \, dt}{(t^2 - y^2)^{1/2} (x^2 - w^2)^{1/2}} \qquad \ldots(3.12)$$

where

$$L(y, w) = \int_0^\infty \xi H(\xi) J_0(\xi y) J_0(\xi w) \, d\xi. \qquad \ldots(3.13)$$

The integrand in (3.13) has no poles, it has only branch points at $\xi = k_1$, $\xi = k_2$. The infinite integral in (3.13) can be converted into an integral with finite limits by the procedure given by Mal (1970).

Suppose that $w > y$ and let complex plane be $\xi = \sigma' + i\tau'$. Further let

$$I = \int_{k_1}^{\infty} M(\xi, \beta_2, \beta_1)\, J_0(\xi y)\, J_0(\xi w)\, d\xi \qquad \ldots(3.14)$$

where

$$M(\xi, \beta_2, \beta_1) = H(\xi). \qquad \ldots(3.15)$$

Consider the integral

$$I_1 = \oint_{C_1} M(\xi, \beta_2, \beta_1)\, J_0(\xi y)\, H_0^{(1)}(\xi w)\, d\xi \qquad \ldots(3.16)$$

around a contour C_1 in the upper right-hand quadrant passing over the branch points $\xi = k_2$, $\xi = k_1$ indented by semicircles of radius δ and ϵ respectively and also consider the integral

$$I_2 = \oint_{C_2} M(\xi, \beta_2, \beta_1)\, J_0(\xi y)\, H_0^{(2)}(\xi w)\, d\xi \qquad \ldots(3.17)$$

around a contour C_2 in the lower right-hand quadrant and passing under the branch points $\xi = k_2$, $\xi = k_1$ indented by semicircles of radius δ and ϵ respectively. There are no singularities within these contours and the integrands in (3.16) and (3.17) satisfy Jordan's Lemma on the infinite quarter circles. If we let $\delta, \epsilon \to 0$ and $R \to \infty$,

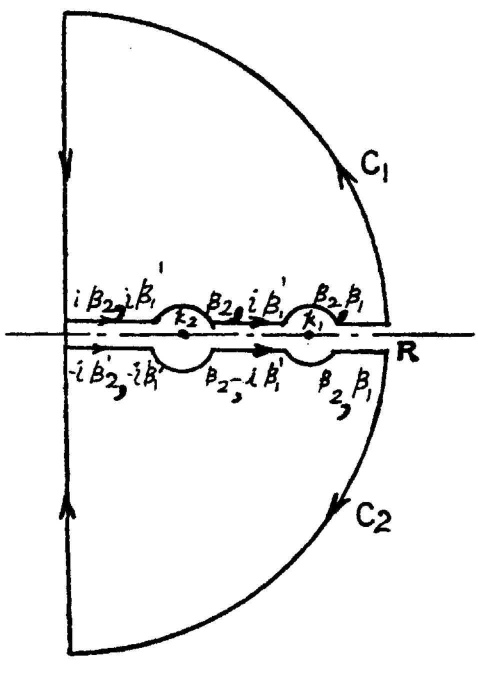

FIG. 1. The contours of integration for the integrals in (3.16) and (3.17).

the contribution from the corresponding arcs tend to zero. Moreover the integrals along the imaginary axis in the upper and lower half planes have values equal in magnitude but opposite in sign, hence they cancel each other. The final result of contour integration along C_1 and C_2 is:

$$I = -\tfrac{1}{2}\int_0^{k_2} \xi J_0(\xi y)\,[M(\xi, i\beta_2', i\beta_1')\,H_0^{(1)}(\xi w)$$

$$+ M(\xi, -i\beta_2', -i\beta_1')\,H_0^{(2)}(\xi w)]\,d\xi$$

$$-\tfrac{1}{2}\int_{k_2}^{k_1} \xi J_0(\xi y)\,[M(\xi, \beta_2, i\beta_1')\,H_0^{(1)}(\xi w)$$

$$+ M(\xi, \beta_2, -i\beta_1')\,H_0^{(2)}(\xi w)]\,d\xi,\; w > y \qquad ...(3.18)$$

where

$$\beta_j' = (k_j^2 - \xi^2)^{1/2}, \quad j = 1, 2. \qquad ...(3.19)$$

Adding the term $\int_0^{k_1} \xi M(\xi, \beta_2, \beta_1)\,J_0(\xi y)\,J_0(\xi w)\,d\xi$, on both the sides of (3.18) and substituting the value of $M(\xi, \beta_2, \beta_1)$ from (3.15) and using (3.14) we get

$$L(y, w) = -i(\mu_1 + \mu_2)\,k_1^2$$

$$\times \int_0^{\gamma} \frac{J_0(\xi k_1 y)\,H_0^{(1)}(\xi k_1 w)\,(\gamma^2 - \xi^2)^{1/2}\,(1 - \xi^2)^{1/2}}{\mu_1(1 - \xi^2)^{1/2} + \mu_2(\gamma^2 - \xi^2)^{1/2}}\,d\xi$$

$$- i\mu_2(\mu_1 + \mu_2)\,k_1^2$$

$$\times \int_{\gamma}^{1} \frac{J_0(\xi k_1 y)\,H_0^{(1)}(\xi k_1 w)\,(\xi^2 - \gamma^2)\,(1 - \xi^2)^{1/2}}{\mu_1^2(1 - \xi^2) + \mu_2^2(\xi^2 - \gamma^2)}\,d\xi \qquad ...(3.20)$$

where $\gamma = k_2/k_1 < 1$.

In case $k_2/k_1 > 1$, we obtain following the above procedure

$$L(y, w) = -i(\mu_1 + \mu_2)\,k_2^2$$

$$\times \int_0^{\gamma_1} \frac{J_0(\xi k_2 y)\,H_0^{(1)}(\xi k_2 w)\,(\gamma_1^2 - \xi^2)^{1/2}\,(1 - \xi^2)^{1/2}}{\mu_1(\gamma_1^2 - \xi^2)^{1/2} + \mu_2(1 - \xi^2)^{1/2}}\,d\xi$$

$$- i\mu_1(\mu_1 + \mu_2)\,k_2^2 \times$$

(*equation continued on p.* 248)

$$\times \int_{\gamma_1}^{1} \frac{J_0(\xi k_2 y)\, H_0^{(1)}(\xi k_2 w)\,(\xi^2 - \gamma_1^2)\,(1 - \xi^2)^{1/2}}{\mu_1^2(\xi^2 - \gamma_1^2) + \mu_2^2(1 - \xi^2)}\, d\xi \qquad \ldots(3.21)$$

where $\gamma_1 = k_1/k_2 < 1$.

The value of this kernel for $w < y$ is obtained by interchanging w and y.

4. Solution of the Integral Equation

The triple integral eqn. (3.4 – 3.5) has been reduced to the integro-differential eqn. (3.12) with the kernel $L(y, w)$ given by (3.20) and (3.21). This form of the kernel is suitable for expansion in powers of k_1 or k_2. Hence an iterative solution of (3.12), valid only for smaller values of the dimensionless frequencies k_1 or k_2, can be obtained by using the method of Jain and Kanwal (1972).

For small values of the argument the Bessel functions $J_0(x)$ and $H_0^{(1)}(x)$ may be expanded in ascending powers of x as

$$J_0(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n},\quad H_0^{(1)}(x) = \left[1 + \frac{2i}{\pi}\log\frac{x}{2}\right] J_0(x) + i\sum_{n=0}^{\infty} b_{2n} x^{2n}$$

where $a_0 = 1$ and the values of the other coefficients a_{2n} and b_{2n} are given by Jain and Kanwal (1972). Using the above expressions in (3.20), $L(y, w)$ can be expressed, for $k_2 < k_1$, as

$$L(y, w) = k_1^2 \log k_1 L_1(y, w) + k_1^2 L_2(y, w) + (k_1^2 \log k_1)^2 L_3(y, w) + k_1^4 \log k_1 L_4(y, w) + O(k_1^4) \qquad \ldots(4.1)$$

where

$$L_1(y, w) = 2M_0/\pi,\quad L_2(y, w) = 2N_0\pi^{-1} + \left(b_0 - i + \frac{2}{\pi}\log\frac{w}{2}\right) M_0$$

$$L_3(y, w) = 0,\quad L_4(y, w) = 2a_2 N_2(y^2 + w^2)/\pi,\quad w > y$$

$$M_{2n} = \int_0^{\gamma} \alpha(\xi)\, d\xi + \int_{\gamma}^{1} \beta(\xi)\, d\xi \qquad \ldots(4.2)$$

$$N_{2n} = \int_0^{\gamma} \alpha(\xi) \log\xi\, d\xi + \int_{\gamma}^{1} \beta(\xi) \log\xi\, d\xi,\quad n = 0, 1, 2 \qquad \ldots(4.3)$$

$$\alpha(\xi) = \frac{(\mu_1 + \mu_2)\, \xi^{2n}(\gamma^2 - \xi^2)^{1/2}\,(1 - \xi^2)}{\mu_1(1 - \xi^2)^{1/2} + \mu_2(\gamma^2 - \xi^2)^{1/2}}$$

$$\beta(\xi) = \frac{\mu_2(\mu_1 + \mu_2)\, \xi^{2n}(\xi^2 - \gamma^2)\,(1 - \xi^2)^{1/2}}{\mu_1^2(1 - \xi^2) + \mu_2^2(\xi^2 - \gamma^2)}$$

Similarly, for $k_1 < k_2$, the expansion of $L(y, w)$ can be obtained in power of k_2 by using the expression (3.21). We shall however give the results for $k_2 < k_1$ only. Next, let us assume the solution of (3.12) in the form

$$h(t^2) = h_0(t^2) + k_1^2 \log k_1 h_1(t^2) + k_1^2 h_2(t^2) + (k_1^2 \log k_1)^2 h_3(t^2) + O(k_1^4 \log k_1). \qquad ...(4.4)$$

Substituting this expansion as well as the expansion (4.1) of $L(y, w)$ in eqns. (3.12), (3.9) and equating the coefficients of equal powers of k_1, we obtain the following equation for the determination of the unknown functions $h_j(t^2)$, $j = 0, 1, 2, \ldots$:

$$\int_c^1 \frac{t h_0(t^2)\, dt}{t^2 - x^2} = \frac{p_0(\mu_1 + \mu_2)}{2\pi\mu_1}; \quad c \leqslant |x| \leqslant 1, \quad \int_c^1 h_0(t^2)\, dt = 0 \qquad ...(4.5)$$

$$\int_c^1 \frac{t h_1(t^2)\, dt}{t^2 - x^2} = -\frac{d}{dx}\int_c^1 h_0(t^2) \int_0^t \int_0^x \frac{L_1(y, w)\, yw\, dy \, . dw}{(x^2 - w^2)^{1/2}(t^2 - y^2)^{1/2}}\, dt; \qquad ...(4.6)$$

$$c \leqslant |x| \leqslant 1, \quad \int_c^1 h_1(t^2)\, dt = 0$$

$$\int_c^1 \frac{t h_2(t^2)\, dt}{t^2 - x^2} = -\frac{d}{dx}\int_c^1 h_0(t^2) \int_0^t \int_0^x \frac{yw L_2(y, w)\, dy\, dw}{(x^2 - w^2)^{1/2}(t^2 - y^2)^{1/2}}\, dt; \qquad ...(4.7)$$

$$c \leqslant |x| \leqslant 1, \quad \int_c^1 h_2(t^2)\, dt = 0$$

$$\int_c^1 \frac{t h_3(t^2)\, dt}{t^2 - x^2} = -\frac{d}{dx}\int_c^1 \int_0^t \int_0^x \frac{yw\,[h_1(t^2)\, L_1(y, w) + h_0(t^2)\, L_3(y, w)]}{(x^2 - w^2)^{1/2}(t^2 - y^2)^{1/2}}\, dy\, dw\, dt; \qquad ...(4.8)$$

$$c \leqslant |x| \leqslant 1, \quad \int_c^1 h_3(t^2)\, dt = 0$$

and so on.

It has been shown by Srivastava and Lowengrub (1968) that the solution of the integral equation

$$\int_c^1 \frac{t h(t^2)\, dt}{t^2 - x^2} = \frac{\pi}{2} f(x); \quad c \leqslant |x| \leqslant 1, \quad \int_c^1 h(t^2)\, dt = 0 \qquad ...(4.9)$$

is given by

$$h(t^2) = \frac{2}{\pi F(t^2 - c^2)^{1/2}(1 - t^2)^{1/2}}$$

$$\times \int_c^1 \left(\frac{1 - x^2}{x^2 - c^2}\right)^{1/2} xf(x)\,dx \int_c^1 \left(\frac{t^2 - c^2}{1 - t^2}\right)^{1/2} \frac{dt}{x^2 - t^2}$$

$$- \frac{2}{\pi}\left(\frac{t^2 - c^2}{1 - t^2}\right)^{1/2} \int_c^1 \left(\frac{1 - x^2}{x^2 - c^2}\right)^{1/2} \frac{xf(x)}{x^2 - t^2}\,dx \qquad \text{...(4.10)}$$

where $F = F[\frac{1}{2}\pi\ , (1 - c^2)^{1/2}]$ is the elliptic integral of the first kind. With the help of relations (4.9) – (4.10) we obtain the solutions of (4.5) – (4.8) as

$$h_0(t^2) = \frac{p_0(\mu_1 + \mu_2)}{\pi\mu_1} \frac{(t^2 - EF^{-1})}{\{(t^2 - c^2)(1 - t^2)\}^{1/2}} \qquad \text{...(4.11)}$$

$$h_1(t^2) = -\frac{p_0 M_0(\mu_1 + \mu_2)}{\pi^2\mu_1} \frac{(t^2 - EF^{-1})(1 + c^2 - 2EF^{-1})}{\{(t^2 - c^2)(1 - t^2)\}^{1/2}} \qquad \text{...(4.12)}$$

$$h_2(t^2) = \frac{p_0(\mu_1 + \mu_2)}{\pi\mu_1\{(t^2 - c^2)(1 - t^2)\}^{1/2}} [d_0(t^2 - EF^{-1})$$

$$- (M_0/\pi)\{(t^2 - c^2)(t^2 - \tfrac{1}{2}(1 - c^2)) - \tfrac{1}{6}EF(1 + c^2)$$

$$+ (\tfrac{1}{2}c^4 - \tfrac{1}{6}c^6) + 2(t^2 - EF^{-1})(1 - EF^{-1})$$

$$- 2(1 - c^2)\,IF^{-2}\} + 2t^2(t^2 - c^2)\,F^{-1}$$

$$\times \{\Pi(\tfrac{1}{2}\pi, (1 - c^2)(1 - t^2)^{-1}, (1 - c^2)^{1/2}) - 1\}] \qquad \text{...(4.13)}$$

$$h_3(t^2) = -\frac{p_0(\mu_1 + \mu_2)}{\pi\mu_1\{(t^2 - c^2)(1 - t^2)\}^{1/2}} \left(\frac{M_0}{\pi}\right)^2 \left(t^2 - \frac{E}{F}\right) \left\{1 + c^2 - \frac{2E}{F}\right\}^2 \qquad \text{...(4.14)}$$

where $E = E[\frac{1}{2}\pi, (1 - c^2)^{1/2}]$ and $\Pi\{(\frac{1}{2}\pi), (1 - c^2)(1 - t^2)^{-1}, (1 - c^2)^{1/2}\}$ are elliptic integrals of the second and third kinds respectively. Using relations (4.11) – (4.14), eqn. (4.4) yields

$$h(t^2) = \frac{p_0(\mu_1 + \mu_2)}{\pi\mu_1}\Big[\{(t^2 - c^2)(1 - t^2)\}^{-1/2}\{(t^2 - EF^{-1})S_1 + S_2\}$$

$$+ t^2\left(\frac{t^2 - c^2}{1 - t^2}\right)^{1/2}\left(\frac{k_1^2 c_1}{2} - 1\right)$$

$$+ \left(\frac{t^2 - c^2}{1 - t^2}\right)\frac{k_1^2 c_1(1 - c^2)}{4} - \frac{t^2}{F}\,\Pi(t^2 - c^2)(1 - t^2)^{-1}\Big] \qquad \text{...(4.15)}$$

The value of σ_{yz}, using the value of $h(t^2)$ from (4.15), comes out to be

$$\sigma_{yz} = -\frac{2p_0}{\pi}\Big[-\frac{\pi}{4}c^2(3-c^2)\left(\frac{k_1^2c_1}{2}-1\right)+\frac{\pi}{2}\Big\{(x^2-c^2)\left(\frac{k_1^2c_1}{2}-1\right)$$

$$+S_1+\frac{k_1^2c_1(1-c^2)}{4}\Big\}+\frac{\pi}{2}\Big\{\left(\left(\frac{k_1^2c_1}{2}-1\right)\right.$$

$$\times x^2(x^2-c^2)+S_1\left(x^2-\frac{E}{F}\right)$$

$$+k_1^2S_2\frac{k_1^2c_1}{4}(1-c^2)(x^2-c^2)\Big\}\{(x^2-1)(x^2-c^2)\}^{-1/2}\Big]$$

$$+\frac{2p_0}{\pi F}\Big[\Big\{\tfrac{1}{6}(1-c^6)-\tfrac{1}{4}(1+c^2)(1-c^4)+\frac{c^2}{3}(1-c^2)\Big\}F$$

$$+\frac{(1-c^2)(2-c^2)}{6}+T_c\Big] \qquad \ldots(4.16)$$

where S_1, S_2, d_0, T_c have been defined in the Appendix.

The two stress intensity factors are given by

$$K_1 = \lim_{x\to 1+} (a)^{1/2}[(x-1)^{1/2}\sigma_{yz}(x,0)]_{x>1}$$

and

$$K_2 = \lim_{x\to c-} (a)^{1/2}[(c-x)^{1/2}\sigma_{yz}(x,0)]_{x<c}.$$

The above relations, with the help of eqn. (4.16), yield

$$K_1 = -\frac{p_0(a)^{1/2}}{\{2(1-c^2)\}^{1/2}}\Big[\left(\left(\frac{k_1^2c_1}{2}\right)-1\right)(1-c^2)+S_1\left(1-\frac{E}{F}\right)$$

$$+k_1^2S_2+\frac{k_1^2c_1}{4}(1-c^2)^2\Big] \qquad \ldots(4.17)$$

and

$$K_2 = -\frac{p_0(a)^{1/2}}{\{2c(1-c^2)\}^{1/2}}\Big[S_1\left(c^2-\frac{E}{F}\right)+S_2k_1^2\Big]. \qquad \ldots(4.18)$$

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APPENDIX

$$S_1 = 1 - \frac{M_0}{\pi} \cdot \left(1 + c^2 - \frac{2E}{F}\right) k_1^2 \log k_1 \times \left\{1 + \frac{M_0}{\pi} k_1^2 \log k_1 \left(1 + c^2 - \frac{2E}{F}\right)\right\} + k_1^2 \left(1 - \frac{E}{F}\right) + k_1^2 d_0$$

$$S_2 = \frac{c_1 EF}{12} (1 + c^2) + \frac{c^4}{2} - \frac{c^6}{6} + \frac{(1 - c^2)}{F^2}$$

$$d_0 = -\frac{c_2}{2}\left[1 + c^2 - \frac{2E}{F}\right] - \frac{c_1}{2}\left[(1 + c^2) \log (1 - c^2)^{1/2} - \frac{2E}{F} \log \left(\frac{1 - c^2}{e^2}\right)^{1/2}\right]$$

$$c_2 = \frac{2N_0}{\pi} + (b_0 - i) M_0 - \frac{2M_0}{\pi} \log 2, \quad c_1 = \frac{2M_0}{\pi}$$

$$T_c = \int_0^{\pi/2} \log \left(1 + \frac{\sqrt{c} - 1}{R^2 - \sqrt{c}}\right) \left\{\frac{R^5}{2} - \frac{R^3}{2} (1 + c^2) + \frac{R}{2} c^2\right\} d\alpha$$

$$R^2 = [1 - (1 - c^2) \sin^2 \alpha]$$

$$I = \int_0^{\pi/2} \sin^2 \theta \, (\sin^2 \theta + c^2 \cos^2 \theta)^{1/2} \{\Pi (\tfrac{1}{2}\pi, \sec^2 \theta, \sqrt{(1 - c^2)}) - F\} \, d\theta.$$