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CONFORMALLY FLAT SPHERICALLY SYMMETRIC CHARGED PERFECT FLUID DISTRIBUTION IN GENERAL RELATIVITY

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A solution of Einstein's field equations representing spherically symmetric charged perfect fluid distribution, which are conformally flat, is obtained. Various physical properties of the model are also discussed.

1. INTRODUCTION

By virtue of the equality of conformal curvature tensors of two conformally related spaces it is clear that, corresponding to the existing physical systems, one can also generate others of considerable interest by introducing a conformal transformation. A considerable interest has been shown to the study of physical properties of space-times which are conformal to certain well known gravitational fields. A number of conformally flat physically significant space-times are known like Schwarzschild internal solution and Lemaitre cosmological universe. Singh and Roy (1966) have investigated the possibilities of existence of electromagnetic fields conformal to some empty space times. Recently a solution of Einstein's field equations representing spherically symmetric perfect fluid distribution which is conformally flat has been obtained by Roy and Raj Bali (1978) which is a generalisation of the solution previously obtained by Singh and Abdussattar (1974).

In this paper we obtain a general solution of Einstein's field equations for distribution of charged perfect fluid with spherical symmetry which is conformally flat. The resulting model is found to be expanding but non-rotating and non-shearing.

2. FIELD EQUATIONS

Einstein-Maxwell's equations for a distribution of charged perfect fluid are

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -8\pi T_{ij} \qquad \ldots(1)$$

$$T_{ij} = (\epsilon + p)\, v_i v_j + p g_{ij} + E_{ij} \qquad \ldots(2)$$

$$E_{ij} = \frac{1}{4\pi} \left[F_{ai} F_{bj} g^{ab} - \tfrac{1}{4} g_{ij} F_{ab} F^{ab}\right] \qquad \ldots(3)$$

where ϵ and p are density and pressure of the fluid. The electromagnetic field tensor F_{ij} satisfies

$$F^{ij}_{;j} = 4\pi\rho v^i \qquad ...(4)$$

$$F_{[ij;k]} = 0 \qquad ...(5)$$

ρ being the current density. The flow vector v^i satisfies

$$g_{ij}v^iv^j = -1 \qquad ...(6)$$

Here and henceforth, a comma and a semicolon denote ordinary and co-variant differential respectively.

We now consider the conformal metric in spherical polar coordinates.

$$ds^2 = e^\lambda(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 - dt^2) \qquad ...(7)$$

where λ is a function of r and t alone. We number the coordinates as $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = t$.

If we take F^{14} as the only non-vanishing component of F^{ij} and $v^i = (v^1, 0, 0, v^4)$, field eqns. (1) for the line-element (7) are

$$\frac{3\lambda_1^2}{4} + \frac{2\lambda_1}{r} - \lambda_{44} - \frac{\lambda_4^2}{4} = 8\pi\,[(\epsilon + p)\,v_1^2 + pe^\lambda] - e^{-\lambda}(F_{14})^2 \qquad ...(8)$$

$$\lambda_{11} + \frac{\lambda_1^2}{4} + \frac{\lambda_1}{r} - \lambda_{44} - \frac{\lambda_4^2}{4} = 8\pi pe^\lambda + e^{-\lambda}(F_{14})^2 \qquad ...(9)$$

$$\frac{3\lambda_4^2}{4} - \lambda_{11} - \frac{\lambda_1^2}{4} - \frac{2\lambda_1}{r} = 8\pi\,[(\epsilon + p)\,v_4^2 - pe^\lambda] + e^{-\lambda}(F_{14})^2 \qquad ...(10)$$

$$\frac{\lambda_1\lambda_4}{2} - \lambda_{14} = 8\pi(\epsilon + p)\,v_1v_4. \qquad ...(11)$$

Equation (6) gives

$$v_4^2 - v_1^2 = e^\lambda. \qquad ...(12)$$

3. Solution of the Field Equations

From eqns. (8) and (9) we have

$$8\pi\,[(\epsilon + p)\,v_1^2] - 2e^{-\lambda}(F_{14})^2 = \frac{\lambda_1^2}{2} + \frac{\lambda_1}{r} - \lambda_{11}. \qquad ...(13)$$

Also, eqns. (9) and (10) readily give

$$8\pi\,[(\epsilon + p)\,v_4^2] + 2e^{-\lambda}(F_{14})^2 = \frac{\lambda_4^2}{2} - \frac{\lambda_1}{r} - \lambda_{44}. \qquad ...(14)$$

Combining eqns. (12), (13) and (14) we obtain

$$8\pi\,[(\epsilon + p)\,e^\lambda] + 4e^{-\lambda}(F_{14})^2 = \frac{\lambda_4^2}{2} - \frac{\lambda_1^2}{2} - \frac{2\lambda_1}{r} - \lambda_{44} + \lambda_{11}. \qquad ...(15)$$

Equations (9) and (15) together give

$$8\pi\epsilon e^{\lambda} + 3e^{-\lambda}(F_{14})^2 = \frac{3}{4}\left(\lambda_4^2 - \lambda_1^2 - \frac{4\lambda_1}{r}\right). \quad ...(16)$$

It is difficult to solve these equations in general. If we take $v_1 = 0$, eqn. (13) gives

$$-e^{-\lambda}(F_{14})^2 = \frac{\lambda_1^2}{4} + \frac{\lambda_1}{2r} - \frac{\lambda_{11}}{2}. \quad ...(17)$$

From (11) we obtain

$$2\lambda_{14} - \lambda_1\lambda_4 = 0. \quad ...(18)$$

The general solution of (18) is

$$e^{\lambda} = [\alpha(r) + \beta(t)]^{-2} \quad ...(19)$$

where α and β are functions of r and t respectively. Hence the metric (7) reduces to the form

$$ds^2 = \frac{1}{(\alpha + \beta)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - dt^2) \quad ...(20)$$

which is the model for a distribution of charged perfect fluid with the flow vector in t-direction.

4. Some Physical Properties

The pressure and density for the model (20) are given by

$$8\pi p = 3(\alpha_1^2 - \beta_4^2) + (\alpha + \beta)\left(2\beta_{44} - \alpha_{11} - \frac{3\alpha_1}{r}\right) \quad ...(21)$$

$$8\pi\epsilon = 3(\beta_4^2 - \alpha_1^2) + 3(\alpha + \beta)\left(\alpha_{11} + \frac{\alpha_1}{r}\right). \quad ...(22)$$

The non-vanishing component of the flow vector, v^4 is given by

$$v_4 = (\alpha + \beta)^{-1}. \quad ...(23)$$

The reality conditions $\epsilon + p > 0$ and $\epsilon + 3p > 0$ lead to

$$\beta_{44} + \alpha_{11} + \frac{\alpha_1}{r} > 0 \quad ...(24)$$

$$(\alpha_1^2 - \beta_4^2) + (\alpha + \beta)\left(\beta_{44} - \frac{\alpha_1}{r}\right) > 0. \quad ...(25)$$

Equation (17) gives

$$F_{14} = (\alpha + \beta)^{-3/2}\left(\frac{\alpha_1}{r} - \alpha_{11}\right)^{1/2}. \quad ...(26)$$

From (4) and (26) the current density ρ is given by

$$\rho = -\frac{(\alpha+\beta)^3}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{\alpha_1}{r}-\alpha_{11}\right)^{1/2}\Big/(\alpha+\beta)^{3/2}\right] \qquad ...(27)$$

The non-vanishing component of the acceleration vector

$\dot{v}_i = v_{i;j}v^j$ is

$$\dot{v}_1 = -\frac{\alpha_1}{(\alpha+\beta)}. \qquad ...(28)$$

Thus the acceleration is always directed in radial direction and the fluid flow in t-direction is uniform. If $\alpha_1 < 0$ acceleration is positive and if $\alpha_1 > 0$, there will be deceleration.

The expression for expansion Φ, rotation and shear tensors are (Ellis 1971)

$$\Phi = v^i_{;i}$$

$$-\omega_{ij} = \tfrac{1}{2}(v_{i;j} - v_{j;i}) + \tfrac{1}{2}(\dot{v}_i v_j - \dot{v}_j v_i)$$

$$\sigma_{ij} = \tfrac{1}{2}(v_{i;j} + v_{j;i}) + \tfrac{1}{2}(\dot{v}_i v_j + \dot{v}_j v_i) - \frac{\Phi}{3}(g_{ij} + v_i v_j).$$

Here we find that $\Phi = 3\beta_4$. Thus the expansion is time-dependent only. All the components of ω_{ij} and σ_{ij} are zero.

Hence the model (20) representing a distribution of charged perfect fluid is expanding with time but non-rotating and non-shearing.

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