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ON THE NULL EINSTEIN-MAXWELL FIELDS IN GENERAL RELATIVITY

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The basic system consisting of Einstein field equations and Maxwell equations for the null electromagnetic field has been studied using the invariant index-free notation. For the compatibility of Einstein-Maxwell equations with certain killing vector fields with fundamental velocity the necessary and sufficient conditions are given.

1. INTRODUCTION

Let V_4 be the space-time of general relativity with the gravitational symmetric tensor field g. Let F be a (1, 1) tensor field such that the antisymmetric electromagnetic field tensor F' of type (0, 2) is given by

$$g(F(X), Y) = F'(X, Y) \qquad \text{...(1.1}a\text{)}$$

and let

$$4K = \text{trace}\,(F^2), \quad k = \det\,(F), \quad D = K^2 - k. \qquad \text{...(1.1}b\text{)}$$

The characteristic equation for F is given by

$$F^4 + 2KF^2 + kI_4 = 0 \qquad \text{...(1.1}c\text{)}$$

where I_4 being the identity transformation on V_4. The (1, 1) tensor field F has been classified into four classes by Mishra (1976) according to its eigen values. An electromagnetic field of the third class is the null electromagnetic field.

For the null electromagnetic fields, all the roots of eqn. (1.1c) of F are equal to zero. However, it is known that (cf. Hlavaty 1958, Mishra 1976) we can choose a set of four linearly independent null vectors $\{e_\alpha\}$ and a set of 1-forms $\{e^\alpha\}$ dual to $\{e_\alpha\}$ which give rise to a non-holonomic frame such that e_1, e_2 are complex conjugates null vectors and e_3, e_4 are real null vectors. Note that e_4 is the only eigen vector belonging to the single eigenvalue 0 and is known as propagation vector. The sets $\{e_\alpha\}$ and $\{e^\alpha\}$ satisfy the following relations:

$$\sqrt{2}\,[F(X)] = e^3(X)\,\{e_1 + e_2\} + \{e^1(X) + e^2(X)\}\,e_4 \qquad \text{...(1.2)}$$

where $X = e^\alpha(X)\,e_\alpha$, and

$$\sqrt{2}\,Fe_1 = \sqrt{2}\,Fe_2 = e_4, \quad \sqrt{2}\,Fe_3 = e_1 + e_2$$

$$F^2(X) = e^3(X)\, e_4, \quad Fe_4 = 0, \quad e^3F = 0, \quad F^3 = 0.$$

Also

$$\left.\begin{aligned} &g(X, Y) = e^1(X)\, e^2(Y) + e^1(Y)\, e^2(X) - e^3(X)\, e^4(Y) - e^3(Y)\, e^4(X) \\ &g(e_1, X) = e^2(X), \quad g(e_2, X) = e^1(X) \\ &g(e_3, X) = -e^4(X), \quad g(e_4, X) = -e^3(X). \end{aligned}\right\} \quad ...(1.3)$$

Definition — A space-time V_4 satisfying (1.1) - (1.3) is said to be endowed with an $(F, g, e_\alpha, e^\alpha)$-structure (of null kind).

The stress tensor of type (1, 1) for $(F, g, e_\alpha, e^\alpha)$-structure is given by

$$T(X) = F^2(X). \quad ...(1.4)$$

The basic system of Einstein-Maxwell equations are assumed to be in the form

$$R(X) = \mu\eta(X)\xi - T(X) + (p + \tfrac{1}{2} R)\,(X) \quad ...(1.5a)$$

where $\quad g(R(X), Y) = \text{Ric}\,(X, Y). \quad ...(1.5b)$

$\mu, p, R(X)$ and R are the density, the pressure, the contracted curvature tensor and the scalar curvature, respectively. ξ is the time-like velocity vector of its orbit such that

$$g(\xi, X) = \eta(X), \; \eta(\xi) = -1. \quad ...(1.5c)$$

Let us choose

$$\xi = \frac{1}{\sqrt{2}} \left\{ \gamma e_3 + \frac{1}{\gamma} e_4 \right\} \quad ...(1.6a)$$

so that (1.5*c*) is satisfied. From eqns. (1.3), (1.5) and (1.6*a*), we get

$$\eta(X) = -\frac{1}{\sqrt{2}} \left\{ \gamma e^3(X) + \frac{1}{\gamma} e^4(X) \right\} \cdot \quad ...(1.6b)$$

The equation of continuity is

$$\text{div}\,(\mu\xi) = 0. \quad ...(1.7)$$

The Maxwell equations in the idealized frame are given by

$$\left.\begin{aligned} &\text{(i)} \quad (\text{div}\, F)\,(X) = \epsilon\eta(X) \\ &\text{(ii)} \quad dF = 0 \end{aligned}\right\} \quad ...(1.8)$$

where ϵ is the energy and d is the operator of exterior derivative.

In the recent years, Mishra (1976), Duggal (1978) (non-null fields only), Ahsan (1978) and Ahsan-Husain (1980, 1981) have studied the geometric properties of

$(F, g, e_\alpha, e^\alpha)$-structure in some detail. The present paper is another attempt to study the geometry of the $(F, g, e_\alpha, e^\alpha)$-structure and here we have studied the basic system consisting Einstein's field equations and Maxwell equations for the null electromagnetic fields using the invariant index-free notations, and have obtained the necessary and sufficient conditions for the compatibility of the field equations with certain Killing vectors having light velocity.

2. Einstein-Maxwell Equations and Compatibility Conditions

From eqns. (1.2), (1.4) and (1.6), Einstein-Maxwell eqns. (1.5*a*), after simplifactions, become

$$R(X) = -\frac{\mu}{2}\left\{\gamma^2 e^3(X)\, e_3 + \frac{1}{\gamma^2} e^4(X)\, e_4 + e^4(X)\, e_3\right\}$$
$$- e^3(X)\, e_4 \left\{\frac{\mu}{2} + 1\right\} + \left(p + \frac{R}{2}\right)(e^3(X)\, e_3 + e^4(X)\, e_4)$$
$$+ \left(p + \frac{R}{2}\right)(e^1(X)\, e_1 + e^2(X)\, e_2). \qquad ...(2.1)$$

Definition — A vector field U is said to be a Killing vector field if

$$(\nabla_X u)(Y) + (\nabla_Y u)(X) = 0$$

where ∇ is the symbol of covariant derivative.

It is known that (Yano 1970) a Killing vector U has light velocity iff $R(U, Y)=0$. Here, we have Ric (e_3, Y) = Ric $(e_4, Y) = 0$, that is, from (1.5*b*) we conclude that $R(X) = 0$ iff X is a Killing vector with light velocity and as such X is either e_3 or e_4, while for X to be e_1 or e_2, we have

$$\text{Ric}\,(e_1, X) = g(R(e_1), X) = e^3(e_1)\, e^3(X) = -e^3(X)\, g(e^3, e_1).$$

Thus we have the following:

Theorem 2.1 — For a space-time V_4 endowed with an $(F, g, e_\alpha, e^\alpha)$-structure, the system of Einstein-Maxwell field equations $(\mu \neq 0)$ are incompatible with a Killing vector e_α having light velocity.

For the compatibility of the Einstein-Maxwell equations we proceed as follows:

Consider two operators l and $\tilde{l}$ as defined by

$$\gamma l = F^2 + \gamma I_4, \quad \gamma \tilde{l} = -F^2 \qquad ...(2.2a)$$

to the tangent space T_x at point x of V_4, γ is an arbitrary constant and I_4 being the identity operator. Here $l + \tilde{l} = I_4$, $lF = Fl = F$, $\tilde{l}F = F\tilde{l} = 0$, and thus it is possible that there exists two complementary spaces C = Image (F) and $\tilde{C}$ = Image (F^2) = Ker (F) corresponding to l and $\tilde{l}$. Let us assume that there exists a vector field U (or e_4) and a 1-form u (or e^3) of $\tilde{l}$ such that

$$\tilde{l}(X) = u(X)U.$$

Using this and $F\tilde{l} = 0$ in (2.2a), we get

$$\left.\begin{aligned} &F^2(X) + \gamma\, u(X)U = 0 \\ &FU = 0, \quad uF = 0 \\ &\qquad \text{rank } F = 2. \end{aligned}\right\} \qquad \ldots(2.2b)$$

Also, we have $h(U, X) = -\, u(X)$, $h(U, U) = 2e_1(U)\, e_2(U)$. If e_4 is Killing and is such that Ric $(e_4, Y) = 0$. (i.e., e_4 is Killing with light velocity). Consequently, by putting $R(e_4) = 0$ in (2.1), we get

$$(p + \tfrac{1}{2} R)\,(e^1(e_4)\, e_1 + e^2(e_4)\, e_2) = 0, \; (\mu \neq 0)$$

and $e^3(\xi) = 0 = e^4(\xi)$.

Also since $\{e_\alpha\}$ are linearly independent, we, therefore, have

Theorem 2.2 — For a space-time V_4 endowed with an $(F, g, e_\alpha, e^\alpha)$-structure defined by (1.1) – (1.3) and (2.2), the Einstein-Maxwell equations ($\mu \neq 0$) are compatible with the Killing propagation vector having fundamental velocity if and only if $R = -2p$ and $e^3(\xi) = 0 = e^4(\xi)$.

If $\mu = 0$, then

$$Re_4 = (p + \tfrac{1}{2} R)\,(e^1(e_4)\, e_1 + e^2(e_4)\, e_2),$$

$$Re^3 = (p + \tfrac{1}{2} R)\,(e^1(e^3)\, e_1 + e^2(e^3)\, e_2).$$

$$Re^4 = (p + \tfrac{1}{2} R)\,(e^1(e^4)\, e_1 + e^2(e^4)\, e_2).$$

Therefore, we have the following:

Corollary — For a space-time V_4 endowed with an $(F, g, e_\alpha, e^\alpha)$-structure defined by all relations (1.1) – (1.3) and (2.2), the Einstein-Maxwell equations are compatible with e_4 or e^3 and e^4 as Killing vectors having light velocity if and only if $R = -2p$.

Remark: It may be noted that for a V_4 endowed with an $(F, g, e_\alpha, e^\alpha)$-structure, the basic system containing Einstein-Maxwell equations [i.e., eqns. (1.5) and

(1.8)] is compatible with the continuity equation (1.7) if $\xi(p) = 0$, i.e., pressure is constant along the ξ-curves.

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