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HALL EFFECTS ON UNSTEADY HYDROMAGNETIC FLOW

D. R. V. PRASADA RAO AND D. V. KRISHNA
Department of Mathematics, S.V.U.A.P.G. Centre, Anantapur 515003 (*A.P.*)

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Hall effects on the non-torsionally generated unsteady hydromagnetic flow in a semi-infinite expanse of an electrically conducting rotating, viscous fluid bounded by an infinite non-conducting plate has been investigated. Exact solutions of the boundary layer equations are obtained by using the Laplace transform treatment. The structure of the velocity distribution and the associated boundary layer is investigated and the ultimate steady state flow is eventually established through inertial oscillations and the propagation of diffused waves.

1. INTRODUCTION

The MHD flow of an incompressible, viscous, electrically conducting fluid in a parallel plate channel due to the non-torsional oscillatory movement of one or both the boundaries has been studied by Debnath (1972). He has analysed the structure of the unsteady hydromagnetic boundary layer flow generated by the non-torsional oscillations of the disk(s). A critical study has been made of the associated hydromagnetic boundary layers adjacent to the disk(s). He found that the difficulty of the hydromagnetic Stokes and Ekman problems associated with the resonant frequency has been resolved in the hydromagnetic analysis. In these investigations, the effects of Hall currents are not considered. However, in a partially ionised gas there occurs a Hall current (Cowling 1957) when the strength of the impressed magnetic field is very strong. These Hall effects play a significant role in determining the flow features. Sato (1961), Yamanishi (1962), Sherman and Sutton (1962) have discussed the Hall effects on the steady hydromagnetic flow between two parallel plates. These effects in the unsteady cases were discussed by Pop (1971, 1979), Sakhnovskii (1963). Katagiri (1969) has studied the effect of Hall currents on the boundary layer flow past a semi-infinite flat plate. Datta and Jana (1976) have investigated the Hall effects on the oscillatory MHD flow past a flat plate. In a recent paper, Debnath *et al.* (1979) have studied the effects of Hall current on unsteady hydromagnetic flow past a porous plate in a rotating fluid system and the structure of the steady and unsteady flow fields is investigated.

This paper is concerned with the Stokes and Ekman problems in magnetohydrodynamics taking Hall effects into account. An exact solution of the initial value problem is obtained by the Laplace transform technique. Both the steady and the transient components of velocity field are explicitly obtained with their implication. The effects of Hall currents on the hydrodynamic boundary layers and shear stress are discussed.

2. Formulation of the Problem

We consider an incompressible, viscous and electrically conducting fluid bounded by an infinite rigid plate at $z = 0$. Both the fluid and plate are in a state of rigid rotation with uniform angular velocity Ω about z-axis normal to the plate. We consider the hydromagnetic flow induced in the fluid in the presence of a uniform magnetic field of strength H_0 normal to the plate by an elliptic harmonic oscillation of the plate in its own plane.

Taking Hall current into account, the generalised Ohm's law is

$$\mathbf{J} + \frac{\omega_e \tau_e}{H_0} \mathbf{J} \times \mathbf{H} = \sigma(E + \mu_e \mathbf{q} \times \mathbf{H}) \qquad ...(2.1)$$

where $\mathbf{q}$, $\mathbf{H}$, E, $\mathbf{J}$, ω_e, τ_e, σ and μ_e are the velocity vector, the magnetic field intensity vector, the electric field, the current density vector, the cyclotron frequency, the electron collision time, the fluid conductivity and the magnetic permeability respectively. In writing eqn. (2.1) the electron pressure gradient, the ion-slip and the thermo-electric effects are neglected. We also assume that, the electric field $E = 0$ (Meyer 1958). Under these assumptions, eqn. (2.1) gives

$$J_x + mJ_y = \sigma\mu_e H_0 v \qquad ...(2.2)$$

$$J_y - mJ_x = -\sigma\mu_e H_0 u \qquad ...(2.3)$$

where $m = \omega_e \tau_e$ is the Hall parameter.

On solving (2.2) - (2.3) we get

$$J_x = \frac{\sigma\mu_e H_0}{1 + m^2}(v + mu) \qquad ...(2.4)$$

$$J_y = \frac{\sigma\mu_e H_0}{1 + m^2}(mv - u). \qquad ...(2.5)$$

Since the plate is of infinite extent we assume that the velocity components are functions of z and t alone. Using (2.4) - (2.5), the equations of motion with reference to a rotating frame with the assumption of no-imposed pressure gradient are

$$\frac{\partial u}{\partial t} - 2\Omega v = \nu \frac{\partial^2 u}{\partial z^2} + \frac{\sigma\mu_e^2 H_0^2}{\rho(1 + m^2)}(mv - u) \qquad ...(2.6)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = \nu \frac{\partial^2 v}{\partial z^2} - \frac{\sigma\mu_e^2 H_0^2}{\rho(1 + m^2)}(v + mu). \qquad ...(2.7)$$

The boundary conditions are:

$$u + iv = U_0(ae^{i\omega t} + be^{-i\omega t}),\ w = 0 \quad \text{on} \quad z = 0,\ t > 0 \qquad \text{...(2.8)}$$

$$u, v \to 0 \quad \text{as} \quad z \to \infty,\ t > 0. \qquad \text{...(2.9)}$$

Introducing the non-dimensional variables

$$z' = \frac{U_0}{\nu} z, \quad (u', v') = \frac{1}{U_0}(u, v), \quad t' = \frac{U_0^2}{\nu} t, \quad \omega' = \frac{\omega}{\Omega}$$

eqns. (2.6) - (2.7) become (dropping the suffixes)

$$\frac{\partial u}{\partial t} - 2k^2 v = \frac{\partial^2 u}{\partial z^2} + \frac{M^2}{1 + m^2}(mv - u) \qquad \text{...(2.10)}$$

$$\frac{\partial v}{\partial t} + 2k^2 u = \frac{\partial^2 v}{\partial z^2} - \frac{M^2}{1 + m^2}(v + mu) \qquad \text{...(2.11)}$$

where $M = \left(\sigma\mu_e^2 H_0^2 \nu/\rho U_0^2\right)^{1/2}$ is the Hartmann number and $k^2 = \Omega\nu/U_0^2$ is the rotation parameter.

Equations (2.10) and (2.11) can be combined as

$$\frac{\partial \phi}{\partial t} + 2ik^2\phi = \frac{\partial^2 \phi}{\partial z^2} - \frac{M^2(1 + im)}{1 + m^2}\phi \qquad \text{...(2.12)}$$

where $\phi = u + iv$.

The initial and boundary conditions are :

$$\left.\begin{array}{l} \phi = 0 \quad \text{for} \quad z \geqslant 0 \quad \text{and} \quad t \leqslant 0 \\ \phi = ae^{i\omega t} + be^{-i\omega t} \quad \text{for} \quad z = 0,\ t > 0 \\ \phi \to 0 \quad \text{as} \quad z \to \infty,\ t > 0. \end{array}\right\} \qquad \text{...(2.13)}$$

Taking Laplace transform, eqn. (2.12) and the conditions (2.13) reduce to

$$\frac{d^2\bar{\phi}}{dz^2} - N^2\bar{\phi} = 0 \qquad \text{...(2.14)}$$

and

$$\bar{\phi} = \frac{a}{s - i\omega} + \frac{b}{s + i\omega} \quad \text{at} \quad z = 0 \qquad \text{...(2.15)}$$

$$\bar{\phi} \to 0 \quad \text{as} \quad z \to \infty$$

where $$N^2 = 2ik^2 + \frac{M^2(1 + im)}{1 + m^2}.$$

The solution of (2.14) is given by

$$\bar{\phi} = \left(\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right) \exp\{-(s + N^2)^{1/2} z\}. \qquad ...(2.16)$$

Evaluating the Laplace inversion integral we obtain

$$\phi = \frac{a}{2} e^{i\omega t}\left[e^{\lambda_1 z} \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} + \lambda_1\sqrt{t}\right) + e^{-\lambda_1 z} \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} - \lambda_1\sqrt{t}\right)\right]$$

$$+ \frac{b}{2} e^{-i\omega t}\left[e^{\lambda_2 z} z \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} + \lambda_2\sqrt{t}\right)\right.$$

$$\left. + e^{-\lambda_2 z} \operatorname{erfc}\left(\frac{z}{2\sqrt{t}} - \lambda_2\sqrt{t}\right)\right] \qquad ...(2.17)$$

where $\lambda_{1,2}^2 = \frac{M^2}{1 + m^2} + i\left(\frac{mM^2}{1 + m^2} + 2k^2 \pm \omega\right)$.

Taking the asymptotic expansion of erfc (z) with complex argument in the form

$$\operatorname{erfc}(\eta) \sim \frac{e^{-\eta^2}}{\eta\sqrt{\pi}} \quad \text{as} \quad |\eta| \to \infty$$

and erfc $(-\eta) = 2 -$ erfc η, and substituting in (2.17) we obtain for very large t

$$\phi \sim a \exp(i\omega t - (a_1 + ib_1) z) + b \exp\{-(i\omega t + (a_2 + ib_2) z)\}$$

$$- \frac{z[\{(a_1^2 - b_1^2) t - \frac{1}{4} z^2 t^{-1}\} - 2ia_1 b_1 t]}{2(\pi)^{1/2} tD_1} \exp\left\{-\left(\frac{z^2}{4t} + (a_1 + ib_1)^2 t\right)\right\}$$

$$- \frac{z[\{(a_2^2 - b_2^2) t - \frac{1}{4} z^2 t^{-1}\} - 2ia_2 b_2 t]}{2(\pi)^{1/2} tD_2}$$

$$\times \exp\left\{-\left(\frac{z^2}{4t} + (a_2 + ib_2)^2 t\right)\right\}$$

where $\lambda_1 = a_1 + ib_1, \quad \lambda_2 = a_2 + ib_2$

$$D_1 = \left\{(a_1^2 - b_1^2) t - \frac{z^2}{4t}\right\}^2 + 4a_1^2 b_1^2 t^2$$

$$D_2 = \left\{(a_2^2 - b_2^2) t - \frac{z^2}{4t}\right\}^2 + 4a_2^2 b_2^2 t^2$$

$$a_{1,2} = \frac{1}{\sqrt{2}}\left[\left\{\left(\frac{M^2}{1 + m^2}\right)^2 + \left(\frac{mM^2}{1 + m^2} + 2k^2 \pm \omega\right)^2\right\}^{1/2} + \frac{M^2}{1 + m^2}\right]^{1/2}$$

$$b_{1,2} = \frac{1}{\sqrt{2}}\left[\left\{\left(\frac{M^2}{1 + m^2}\right) + \left(\frac{mM^2}{1 + m^2} + 2k^2 \pm \omega\right)^2\right\}^{1/2} - \frac{M^2}{1 + m^2}\right]^{1/2}.$$

The first two terms in (2.18) correspond to the complex velocity of the steady oscillatory flow and the remaining terms represent the time dependent hydromagnetic inertial oscillations in the fluid. These inertial oscillations exponentially decay with time and the ultimate flow consists of persistant oscillations which decay at distances of order $1/a_1$ and $1/a_2$. Thus the ultimate steady oscillating flow consists of two distinct boundary layers of thickness of order $1/a_1$ and $1/a_2$. From the expressions of a_1 and a_2, we observe that the boundary layer thickness increases with increase in Hall parameter m where as it decreases with increase either in Hartmann number M or rotation parameter k^2. Also, for fixed m, M and k^2 the frequency of oscillation of the plate decreases the thickness of the boundary layers and remain bounded for all values of the flow parameters m, M, k^2 and ω. When $\omega = 0$, $a_1 = a_2$ the two layers combine into a single layer of thickness of the order

$$\left[\left\{\left(\frac{M^2}{1+m^2}\right)^2 + \left(\frac{mM^2}{1+m^2} + 2k^2\right)^2\right\}^{1/2} + \frac{M^2}{1+m^2}\right]^{-1/2}.$$

The most remarkable feature of the asymptotic solution is that the second and fourth terms in (2.18) ensure the existence of inertial oscillations. The frequency of these oscillations is $\left[\frac{mM^2}{1+m^2} + 2k^2 + \omega\right]$, which increases due to an increase in either Hartmann number M or roation parameter k^2. With increase in Hall parameter m, the frequency decreases when $m > 1$ and increases the frequency of the inertial oscillations. These oscillations decay exponentially within the above boundary layers.

Finally it follows from the arguments of the complementary error function involved in (2.17) that the two ultimate boundary layers are established through the inertial oscillations in non-dimensional times of the order $(a_r^2 + b_r^2)^{-1}$, $r = 1, 2$. When $\omega = 0$, these times combine into the time of order

$$\left[\left(\frac{M^2}{1+m^2}\right)^2 + \left(\frac{mM^2}{1+m^2} + 2k^2\right)^2\right]^{-1/2}$$

which increases with m and decreases with increase in M or k^2. The results of this analysis are in perfect agreement with those of Debnath *et al.* (1979) provided the effect of suction is neglected.

The non-dimensional components of shear stress are given by

$$\begin{aligned}\tau_x + i\tau_y &= -\left(\frac{\partial \phi}{\partial z}\right)_{z=0} \\ &= ae^{i\omega t}\left[\frac{1}{\sqrt{\pi t}} \exp\{-(a_1+ib_1)^2 t\}\right. \\ &\quad \left. + (a_1+ib_1)\operatorname{erf}(a_1+ib_1)\sqrt{t}\right] + be^{-i\omega t} \\ &\quad \times \left[\frac{1}{\sqrt{\pi t}} \exp\{-(a_2+ib_2)^2 t\} + (a_2+ib_2)\operatorname{erf}(a_2+ib_2)\sqrt{t}\right]. \quad \ldots(2.19)\end{aligned}$$

As $t \to \infty$, $\mathrm{erf}(t) \to 1$ and the shear stress components at the plate for the final Ekman layer with $a = b = 1$ and $\omega t = \pi/2$ is given by

$$\tau_x + i\tau_y = (b_2 - b_1) + i(a_1 - a_2). \qquad \ldots(2.20)$$

This shows that the shear stress components decrease with increase in m where as they decrease due to an increase in either Hartmann number M or rotation parameter k^2. It is also observed that the frequency of oscillation of the plate increases the shear stress components. It is worth nothing that the stress component τ_y is greater than τ_x in magnitude. In case of uniform motion of the plate i.e., $\omega = 0$ the shear stress for the primary flow (τ_x) is always greater than that for the secondary flow (τ_y). Using the expansions of the complementary error function with complex argument (Strand 1965) solution (2.17) can be rewritten in the form

$$\phi(z, t) = \frac{a}{2} \exp i(\omega - 2a_1 b_1) t \,[e^{a_1 z} f(\eta + a_1 \sqrt{t}, b_1 \sqrt{t})$$
$$+ e^{-a_1 z} \bar{f}(\eta - a_1 \sqrt{t}, b_1 \sqrt{t})] + \frac{b}{2} \exp\{- i(\omega + 2a_1 b_1) t\}$$
$$\times [e^{a_2 z} f(\eta + a_2 \sqrt{t}, b_2 \sqrt{t}) + e^{-a_2 z} \bar{f}(\eta - a_2 \sqrt{t}, b_2 \sqrt{t})],$$
$$\text{when } z > \max(2a_1 t, 2a_2 t); \qquad \ldots(2.21)$$

$$\phi(z, t) = a e^{i\omega t} [\exp\{-(a_1 + ib_1) z\} + \tfrac{1}{2} \exp(-2ia_1 b_1 t)$$
$$\times \{e^{a_1 z} f(\eta + a_1 \sqrt{t}, b_1 \sqrt{t}) - e^{-a_1 z} f(a_1 \sqrt{t} - \eta b_1 \sqrt{t})\}]$$
$$+ b e^{-i\omega t} [\exp\{-(a_2 + ib_2) z\} + \tfrac{1}{2} \exp(-2ia_2 b_2 t)$$
$$\times \{e^{a_2 z} f(\eta + a_2 \sqrt{t}, b_2 \sqrt{t}) - e^{-a_2 z} f(a_2 \sqrt{t} - \eta, b_2 \sqrt{t})\}],$$
$$\text{when } z < \min(2a_1 t, 2a_2 \sqrt{t}) \qquad \ldots(2.22)$$

where $\eta = z/2\sqrt{t}$ and $\bar{f}(x, y)$ is the complex conjugate of $f(x, y)$.

These expressions indicate the generation and propogation of hydromagnetic diffused waves travelling away from the plate with phase velocity c given by

$$c = \sqrt{2}\, U_0 \left[\left\{\left(\frac{M^2}{1 + m^2}\right) + \left(\frac{mM^2}{1 + m^2} + 2k^2 + \omega\right)^2\right\}^{1/2} + \frac{M^2}{1 + m^2}\right]^{1/2}$$

which decreases with increase in M or k^2, where as it decreases with increase in m. It is worth noting that the frequency of oscillation of the plate increases the phase velocity of the diffused waves. In fact, (2.22) represents the unsteady flow field behind the wave front $z = ct$, where as (2.21) describes the flow ahead of it. Ultimately, these waves decay within the final boundary layers.

Finally, the existence of the hydromagnetic waves in this work is consistent with the results of Debnath *et al.* (1979) who have also discussed the generation and propogation of these waves in more general configuration.

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