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CLASSES OF GENERAL AXISYMMETRIC SOLUTIONS OF EINSTEIN-MAXWELL EQUATIONS

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An exact solution of the Einstein equations for a stationary axially symmetric distribution of mass composed of all types of multipoles is obtained. Following Ernst (1968), from this vacuum solution the corresponding solution of the coupled Einstein-Maxwell equations is derived. A solution of Einstein-Maxwell fields for a static axially symmetric system composed of all types of multiples is also obtained.

1. INTRODUCTION

Young and Bently (1975) have outlined the explicit form of the metric for a charged dust and mass distribution with monopole plus quadrupole moment. Following the method due to Ernst (1968), Krori and Choudhury (1979) have presented the solutions of static axially symmetric Einstein-Maxwell fields obtained by Young and Bentley. Ernst has derived the metric for the monopole case and has shown that these represent Kerr and Kerr-Newman space-times. Following Ernst (1968), Tomimatsu and Sato (1973) have derived a series of solutions which represent the gravitational fields of spinning masses. Ernst (1973) and Yamazaki (1978) have presented the solutions of Einstein-Maxwell equations for the Tomimatsu-Sato family. But it appears that the solution of stationary axially symmetric Einstein-Maxwell fields for a system composed of all types of multipoles has not been derived. In section 2 and 3 of this paper, this has been done.

Misra *et al.* (1973) outlined a method for generating a class of electromagnetic fields and derived a particular solution of this class. Following the method due to Misra *et al.* (1973) we have presented in section 4 a new solution of Einstein-Maxwell fields for a static axially symmetric system composed of all types of multipoles.

2. EINSTEIN FIELD EQUATIONS AND THEIR SOLUTIONS

The stationary axially symmetric line element may be taken in the form

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2 \qquad \text{...(2.1)}$$

where f, γ and ω are functions of ρ and z. According to Ernst (1968*a*) these functions can be obtained from a complex function ξ_0 which satisfies the following differential equation:

$$(\xi_0\xi_0^* - 1)\,\nabla^2\xi_0 = 2\xi_0^*\,\nabla\xi_0.\nabla\xi_0. \qquad ...(2.2)$$

The relations between the metric functions (f, γ, ω) and ξ_0 are given by

$$f = \mathrm{Re}\,\frac{\xi_0 - 1}{\xi_0 + 1} \qquad ...(2.3)$$

$$\frac{\partial\gamma}{\partial\rho} = \frac{\rho}{(\xi_0\xi_0^* - 1)^2}\left[\frac{\partial\xi_0}{\partial\rho}.\frac{\partial\xi_0^*}{\partial\rho} - \frac{\partial\xi_0}{\partial z}.\frac{\partial\xi_0^*}{\partial z}\right] \qquad ...(2.4)$$

$$\frac{\partial\gamma}{\partial z} = \frac{\rho}{(\xi_0\xi_0^* - 1)^2}\,\mathrm{Re}\left[\frac{\partial\xi_0}{\partial\rho}.\frac{\partial\xi_0^*}{\partial z}\right] \qquad ...(2.5)$$

$$\frac{\partial\omega}{\partial\rho} = -\frac{2\rho}{(\xi_0\xi_0^* - 1)^2}\,\mathrm{Im}\left[(\xi_0^* + 1)^2\frac{\partial\xi_0}{\partial z}\right] \qquad ...(2.6)$$

and

$$\frac{\partial\omega}{\partial z} = \frac{2\rho}{(\xi_0\xi_0^* - 1)^2}\,\mathrm{Im}\left[(\xi_0^* + 1)^2\frac{\partial\xi_0}{\partial\rho}\right]. \qquad ...(2.7)$$

Let us now introduce a new potential ψ such that

$$\xi_0 = -e^{i\alpha}\coth\psi. \qquad ...(2.8)$$

Equation (2.2) shows that the real function ψ statisfies the Laplace equation

$$\nabla^2\psi = 0. \qquad ...(2.9)$$

Consequently one can express ψ in terms of a multipole expansion. Let us now transform ρ and z to prolate spheroidal co-ordinates

$$\rho = (x^2 - 1)^{1/2}(1 - y^2)^{1/2}, \quad z = xy.$$

Then the Laplacian operator takes the form

$$\nabla^2 \equiv \frac{1}{x^2 - y^2}\left[\frac{\partial}{\partial x}\left\{(x^2 - 1)\frac{\partial}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{(1 - y^2)\frac{\partial}{\partial y}\right\}\right]. \qquad ...(2.10)$$

In this co-ordinate system we also have

$$\nabla A.\,\nabla B = \frac{1}{x^2 - y^2}\left[(x^2 - 1)\frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial y} + (1 - y^2)\frac{\partial A}{\partial y}\cdot\frac{\partial B}{\partial y}\right]. \qquad ...(2.11)$$

Then the solution to eqn. (2.9) may be expressed as

$$\psi = \sum_0^\infty \alpha_i Q_i(x)\,P_i(y) \qquad ...(2.12)$$

where $P_i(y)$ and $Q_i(x)$ are Legendre functions of first and second kinds and α_i is a constant.

If α_i is taken as $(2l + 1)$ one gets (Copson 1950)

$$\sum_{0}^{\infty} (2l + 1)\, Q_l(x)\, P_l(y) = \frac{1}{x - y}. \qquad ...(2.13)$$

Equation (2.12) and (2.13) yield

$$\psi = \frac{1}{x - y}. \qquad ...(2.14)$$

Taking $\alpha = 0$, eqns. (2.8) and (2.14) give

$$\xi_0 = \frac{g + 1}{g - 1} \qquad ...(2.15)$$

where $\quad g = \exp\left(2/(y - x)\right). \qquad ...(2.16)$

The symmetry of eqns. (2.10) and (2.11) in x and y shows that if $\xi_0(x, y)$ is a solution to eqn. (2.2) then so is $\xi_0(y, x)$. When this transformation is applied to the solution (2.15) one arrives at a new solution corresponding to

$$\xi_0 = -\frac{g + 1}{g - 1}. \qquad ...(2.17)$$

Then if one looks for a linear combination of the solutions (2.15) and (2.17) with the property that it also satisfies eqn. (2.2) one arrives at the solution

$$\xi_0 = \frac{g + 1}{g - 1}(\cos\lambda - i \sin\lambda). \qquad ...(2.18)$$

The parameter λ may assume any real value.

Now one can easily calculate the metric functions f, ω and γ from ξ_0. Using eqn. (2.3) one gets

$$f = \frac{2g}{(g^2 + 1) + (g^2 - 1)\cos\lambda}. \qquad ...(2.19)$$

Equations (2.4) and (2.5) in terms of ψ can be expressed as

$$\frac{\partial\gamma}{\partial\rho} = \rho\left[\left(\frac{\partial\psi}{\partial\rho}\right)^2 - \left(\frac{\partial\psi}{\partial z}\right)^2\right] \qquad ...(2.20)$$

and

$$\frac{\partial\gamma}{\partial z} = 2\rho\, \frac{\partial\psi}{\partial\rho} \cdot \frac{\partial\psi}{\partial z}. \qquad ...(2.21)$$

In terms of x, y eqns. (2.20) and (2.21) may be written as follows (Copson 1950):

$$\frac{\partial \gamma}{\partial x} = \frac{1-y^2}{x^2-y^2}\left[x(x^2-1)\left(\frac{\partial \psi}{\partial x}\right)^2 - x(1-y^2)\left(\frac{\partial \psi}{\partial y}\right)^2 - 2y(x^2-1)\frac{\partial \psi}{\partial x}\cdot\frac{\partial \psi}{\partial y}\right] \quad ...(2.22)$$

$$\frac{\partial \gamma}{\partial y} = \frac{x^2-1}{x^2-y^2}\left[y(x^2-1)\left(\frac{\partial \psi}{\partial x}\right)^2 - y(1-y^2)\left(\frac{\partial \psi}{\partial y}\right)^2 + 2x(1-y^2)\frac{\partial \psi}{\partial x}\cdot\frac{\partial \psi}{\partial y}\right]. \quad ...(2.23)$$

Solving these equations one gets

$$\gamma = -\tfrac{1}{2}\,\frac{(x^2-1)(1-y^2)}{(x-y)^4}. \quad ...(2.24)$$

The constant of integration is chosen so that $\gamma \to 0$ for large x.

In terms of ψ eqns. (2.6) and (2.7) can be written as

$$\frac{\partial \omega}{\partial \rho} = 2\rho \sin\lambda \,\frac{\partial \psi}{\partial z} \quad ...(2.25)$$

and

$$\frac{\partial \omega}{\partial z} = -2\rho \sin\lambda\,\frac{\partial \psi}{\partial \rho}. \quad ...(2.26)$$

The solution of these equations yields

$$\omega = \frac{2(1-y^2)}{y-x}\sin\lambda. \quad ...(2.27)$$

The constant of integration is chosen so as to get $\omega \to 0$ for large x.

From eqns. (2.19), (2.24) and (2.27) one finds that as $x \to \infty$, $f \to 1$, $\omega \to 0$ and $\gamma \to 0$ which shows that at large distances the space is Minkowskian.

3. Solution of Einstein-Maxwell Equations

The coupled Einstein-Maxwell equations, according to Ernst, can be written in terms of a pair of complex functions ϵ and Φ and are given by

$$f\,\nabla^2\epsilon = \nabla\epsilon.\,\nabla\epsilon + 2\Phi^*(\nabla\Phi.\,\nabla\epsilon) \quad ...(3.1a)$$

$$f\,\nabla^2\Phi = \nabla\epsilon.\,\nabla\Phi + 2\Phi^*(\nabla\Phi.\,\nabla\Phi) \quad ...(3.1b)$$

where the real and imaginary parts of Φ represent the electric and magnetic scalar potentials and

$$f = \tfrac{1}{2}(\epsilon + \epsilon^*) + \varphi\Phi^*. \quad ...(3.2)$$

Applying the transformations

$$\epsilon = \frac{\xi - 1}{\xi + 1} \qquad \text{...(3.3}a\text{)}$$

$$\Phi = \frac{q}{\xi + 1} \qquad \text{...(3.3}b\text{)}$$

into eqn. (3.1*a*) or (3.1*b*) one gets a single complex equation

$$[\xi\xi^* - (1 - qq^*)] \nabla^2\xi = 2\xi^* \nabla\xi . \nabla\xi. \qquad \text{...(3.4)}$$

For the vacuum axially symmetric Einstein-Maxwell field Ernst has replaced ξ by ξ_0 from $\xi = (1 - qq^*)^{1/2} \xi_0$ satisfying eqn. (3.4) where q is a complex constant connected with charge.

Equations (3.2) and (3.3) will now determine the metric coefficients while eqn. (3.3*b*) will determine the electromagnetic potentials. The metric coefficients and the electromagentic potentials are given by

$$f = \frac{4(1 - qq^*) g}{(1 - qq^*)(g + 1)^2 + (g - 1)^2 + 2(1 - qq^*)^{1/2}(g^2 - 1) \cos \lambda} \qquad \text{...(3.5)}$$

$$\gamma = -\tfrac{1}{2} \frac{(x^2 - 1)(1 - y^2)}{(1 - qq^*)(x - y)^4} \qquad \text{...(3.6)}$$

$$\omega = \frac{2 \sin \lambda}{(1 - qq^*)^{1/2}} \cdot \frac{1 - y^2}{y - x} \qquad \text{...(3.7)}$$

$$A_3 = \frac{\{(1 - qq^*)^{1/2}(g + 1) \sin \lambda\} (qq^*)^{1/2}(g - 1)}{(1 - qq^*)(g + 1)^2 + (g - 1)^2 + 2(1 - qq^*)^{1/2}(g^2 - 1) \cos \lambda} \qquad \text{...(3.8)}$$

$$A_4 = \frac{\{(1 - qq^*)^{1/2}(g + 1) \cos \lambda + (g + 1)\} (qq^*)^{1/2}(g - 1)}{(1 - qq^*)(g + 1)^2 + (g - 1)^2 + 2(1 - qq^*)^{1/2}(g^2 - 1) \cos \lambda} \qquad \text{...(3.9)}$$

4. MPST Method for Solution of Einstein-Maxwell Equations

The line element for the static axially symmetric Einstein-Maxwell field case may be written in the Weyl form

$$ds^2 = e^{2u} dt^2 - e^{-2u} [e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] \qquad \text{...(4.1)}$$

where u and γ are functions of ρ and z.

Following Misra *et al.* (1973) the field equations are written as

$$\frac{\partial^2 u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{\rho} \cdot \frac{\partial u}{\partial \rho} = -e^{-2u} \left[\left(\frac{\partial c}{\partial \rho}\right)^2 + \left(\frac{\partial c}{\partial z}\right)^2\right] \qquad \text{...(4.2)}$$

$$\frac{1}{\rho}\cdot\frac{\partial\Upsilon}{\partial\rho}=\left[\left(\frac{\partial u}{\partial\rho}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2\right]+e^{-2u}\left[\left(\frac{\partial c}{\partial\rho}\right)^2-\left(\frac{\partial c}{\partial z}\right)^2\right] \quad ...(4.3a)$$

$$\frac{1}{\rho}\frac{\partial\Upsilon}{\partial z}=2\frac{\partial u}{\partial\rho}\cdot\frac{\partial u}{\partial z}+2\,e^{-2u}\frac{\partial c}{\partial\rho}\cdot\frac{\partial c}{\partial z} \quad ...(4.3b)$$

$$\frac{\partial^2 c}{\partial\rho^2}+\frac{\partial^2 c}{\partial z^2}+\frac{1}{\rho}\cdot\frac{\partial c}{\partial\rho}=2\left(\frac{\partial c}{\partial\rho}\cdot\frac{\partial u}{\partial\rho}+\frac{\partial c}{\partial z}\cdot\frac{\partial u}{\partial z}\right) \quad ...(4.4)$$

where C is the same potential as defined by Misra *et al.* (1973). Now introducing a new complex function E defined by

$$E=e^u+ic \quad ...(4.5)$$

one finds from eqns. (4.2) and (4.4) that E satisfies the equation

$$\frac{\partial^2 E}{\partial\rho^2}+\frac{\partial^2 E}{\partial z^2}+\frac{1}{\rho}\frac{\partial E}{\partial\rho}=e^{-u}\left[\left(\frac{\partial E}{\partial\rho}\right)^2+\left(\frac{\partial E}{\partial z}\right)^2\right]. \quad ...(4.6)$$

Taking the transformation

$$E=\frac{\chi-1}{\chi+1} \quad ...(4.7)$$

eqn. (4.6) can be written as

$$\frac{\partial^2\chi}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial\chi}{\partial\rho}+\frac{\partial^2\chi}{\partial z^2}=\frac{2\chi^*}{\chi\chi^*-1}\left[\left(\frac{\partial\chi}{\partial\rho}\right)^2+\left(\frac{\partial\chi}{\partial z}\right)^2\right]. \quad ...(4.8)$$

Introducing a new potential ψ such that

$$\chi=-e^{i\alpha}\coth\psi \quad ...(4.9)$$

where α is a real constant, one finds that ψ satisfies the Laplace equation

$$\nabla^2\psi=0. \quad ...(4.10)$$

In the usual spheroidal co-ordinates the solution to eqn. (4.10) (as in section 2) may be expressed as

$$\psi=-\frac{1}{y-x}. \quad ...(4.11)$$

Equation (4.9) and (4.11) give

$$\chi=(\cos\alpha+i\sin\alpha)\frac{g+1}{g-1} \quad ...(4.12)$$

where $g=\exp(2/(y-x))$. ...(4.13)

Equations (4.5), (4.7) and (4.12) give

$$e^u=\frac{2g}{(g^2+1)+(g^2-1)\cos\alpha} \quad ...(4.14)$$

and $$C = \frac{(g^2 - 1) \sin \alpha}{(g^2 + 1) + (g^2 - 1) \cos \alpha}. \qquad \ldots(4.15)$$

As x tends to ∞, eqns. (4.14) and (4.15) show that

$$e^u \to 1 \text{ and } c \to 0.$$

In spheroidal coordinates the eqns. (4.3) takes the form

$$\frac{\partial \Upsilon}{\partial x} = \frac{1 - y^2}{x^2 - y^2} \left[x(x^2 - 1) \left(\frac{\partial \psi}{\partial x} \right)^2 - x(1 - y^2) \left(\frac{\partial \psi}{\partial y} \right)^2 - 2y(x^2 - 1) \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right] \qquad \ldots(4.16a)$$

and

$$\frac{\partial \Upsilon}{\partial y} = \frac{x^2 - 1}{x^2 - y^2} \left[y(x^2 - 1) \left(\frac{\partial \psi}{\partial x} \right)^2 - y(1 - y^2) \left(\frac{\partial \psi}{\partial y} \right)^2 + 2x(1 - y^2) \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right] \qquad \ldots(4.16b)$$

Solving these equations we get

$$\Upsilon = -\frac{1}{2} \frac{(x^2 - 1)(1 - y^2)}{(x - y)^4}. \qquad \ldots(4.17)$$

The constant of integration is taken as zero from the boundary condition at infinity.

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