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A PLANE SYMMETRIC NON-STATIC COSMOLOGICAL MODEL IN GENERAL RELATIVITY

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A plane symmetric non-static cosmological model representing perfect fluid distribution has been obtained which is inhomogeneous and anisotropic and a particular case of which is gravitationally radiative.

INTRODUCTION

In recent years cosmological model exhibiting plane symmetry have attracted the attention of various authors. Plane symmetric perfect fluid distribution was first discussed by Taub (1956) in which the flow was taken to be isentropic. As a special case of the plane symmetric cosmological models Bianchi type I space-times have been extensively studied by Heckmann and Schucking (1962), Thorne (1967), Jacobs (1968, 1969), Singh and Singh (1968, 1969), Singh and Abdussattar (1973), Roy and Singh (1976) and Roy and Prakash (1979). These solutions have been obtained under different geometrical and physical conditions such as the *a priori* stipulation of the equation of state or imposition of the Petrov-type conditions. It is a well known fact that the free gravitational field affects the flow of the fluid by inducing shear in the flow lines, (Ellis 1971). It is therefore interesting to study the cosmological models with given Petrov-types. We would expect a realistic cosmological model to contain gravitational radiation apart from a distribution of disordered radiation in thermal equilibrium. A radiative solution will be either Petrov-type II or III. In our previous paper (Roy and Narain 1979) we have obtained a gravitationally degenerate Petrov type I solution. In the present paper we have derived a cosmological model, which has as a sub-case a Petrov type II solution so that it represents a gravitationally radiating cosmological model.

We consider the metric in the form

$$ds^2 = dt^2 - dx^2 - B^2dy^2 - C^2dz^2. \qquad \ldots(1)$$

where the metric potentials are functions of x and t. The enery momentum tensor for the perfect fluid distribution is given by

$$T_{ij} = (\rho + p)\, v_i v_j - p g_{ij} \qquad \ldots(2)$$

together with

$$g^{ij} v_i v_j = 1 \qquad \ldots(3)$$

ρ being the density, p the pressure and v_i the flow vector. The field equations are

$$-8\pi T_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij} + \Lambda g_{ij}. \qquad ...(4)$$

where Λ is the cosmological constant. Equations (2) and (4) for the metric (1) lead to

$$v_2 = v_3 = 0. \qquad ...(5)$$

The field eqns. (4) lead to

$$-8\pi\,[(\rho + p)\,v_1^2 + p] = \left[\frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{B_1C_1 - B_4C_4}{BC}\right] \qquad ...(6)$$

$$-8\pi p = \frac{C_{44} - C_{11}}{C} - \Lambda \qquad ...(7)$$

$$-8\pi p = \frac{B_{44} - B_{11}}{B} - \Lambda \qquad ...(8)$$

$$-8\pi\,[(\rho + p)\,v_4^2 - p] = \left[\frac{B_{11}}{B} + \frac{C_{11}}{C} + \frac{B_1C_1 - B_4C_4}{BC}\right] + \Lambda \qquad ...(9)$$

$$-8\pi\,[\rho + p)\,v_1v_4] = \left[\frac{B_{14}}{B} + \frac{C_{14}}{C}\right]. \qquad ...(10)$$

From eqn. (3), we have

$$v_4^2 - v_1^2 = 1. \qquad ...(11)$$

In the above equations suffixes 1 and 4 after B and C denote partial differentiation with respect to x and t respectively. From eqns. (7) and (8) we have

$$\frac{B_{uv}}{B} = \frac{C_{uv}}{C} \qquad ...(12)$$

where

$$\left.\begin{aligned} u &= \tfrac{1}{2}(x + t) \\ v &= \tfrac{1}{2}(x - t). \end{aligned}\right\} \qquad ...(13)$$

From eqns. (6), (7), (8), (9), (10) and (12), we have

$$\left[\frac{B_{uu} + B_{vv}}{4B} + \frac{C_{uu} + C_{vv}}{4C}\right]^2 - \left[\frac{B_uC_v + B_vC_u}{2BC}\right]^2$$
$$= \left[\frac{B_{uu} - B_{vv}}{4B} + \frac{C_{uu} - C_{vv}}{4C}\right]^2. \qquad ...(14)$$

SOLUTION OF FIELD EQUATIONS

Let us assume that

$$B = \alpha(u)\,.\,\beta(v). \qquad ...(15)$$

and

$$C = f(u) \cdot g(v). \qquad \text{...(16)}$$

From eqns. (12), (15) and (16), we have

$$\frac{(\alpha_u/\alpha)}{(f_u/f)} = \frac{(g_v/g)}{(\beta_v/\beta)} = k_1. \qquad \text{...(17)}$$

where k_1 is an arbitrary constant. Integrating eqn. (17), we get

$$\alpha = Mf^{k_1} \qquad \text{...(18)}$$

$$g = N\beta^{k_1} \qquad \text{...(19)}$$

where M and N are constants of integration. From eqns. (14), (15), (16), (18) and (19), we have $\left[\frac{(f_u/f)}{(f_{uu}/f_u)}\right]$

$$= -\left[\left\{(k_1+1)^2\frac{\beta_{vv}}{\beta} + k_1(k_1^2-1)\frac{\beta_v^2}{\beta^2}\right\}\Big/\left\{k_1(k_1^2-1)\frac{\beta_{vv}}{\beta} + \{k_1^2(k_1^2-1)^2-(k_1^2+1)^2\}\frac{\beta_v^2}{\beta^2}\right\}\right]$$

$$= k_2. \qquad \text{...(20)}$$

where k_2 is an arbitrary constant. Integrating eqn. (20), we get

$$f = (au+b)^{\mu} \qquad \text{...(21)}$$

and

$$\beta = (mv+n)^{\nu} \qquad \text{...(22)}$$

where a, b, m and n are arbitrary constants and

$$\mu = \frac{k_2}{k_2-1}, \quad \nu = \frac{k_1+1}{k_1^2+1} - \frac{k_2}{k_2-1}.$$

By suitable transformation metric (1) reduces to the form

$$ds^2 = dT^2 - dX^2 - \{U\}^{2k_1\mu} \cdot \{V\}^{2\nu} \cdot dY^2 - \{U\}^{2\mu} \cdot \{V\}^{2k_1\nu} \cdot dZ^2. \qquad \text{...(23)}$$

where

$$\left.\begin{aligned} U &= \tfrac{1}{2}(X+T) \\ V &= \tfrac{1}{2}(X-T). \end{aligned}\right\} \qquad \text{...(24)}$$

3. Some Physical and Geometrical Features

The pressure and density for the model (23) are given by

$$8\pi p = -\left[\frac{4k_1\mu\nu}{(T^2-X^2)}\right] + \Lambda \qquad \text{...(25)}$$

and

$$8\pi\rho = \left[\frac{4\mu\nu(k_1^2 + k_1 + 1)}{(T^2 - X^2)}\right] - \Lambda. \qquad \text{...(26)}$$

The non-vanishing components of the flow vector are

$$v_1 = -\frac{X}{(T^2 - X^2)^{1/2}} \qquad \text{...(27)}$$

and

$$v_4 = \frac{T}{(T^2 - X^2)^{1/2}}. \qquad \text{...(28)}$$

Clearly the region of space-time in which this is valid, is $T^2 - X^2 > 0$. The reality conditions $p > 0$ and $\rho > p$ imply that

$$\left[\frac{4k_1\mu\nu}{(T^2 - X^2)}\right] < \Lambda < \left[\frac{2\mu\nu(k_1 + 1)^2}{(T^2 - X^2)}\right]. \qquad \text{...(29)}$$

The vector

$$\dot{v}_i = v_{i;j}v^j = 0. \qquad \text{...(30)}$$

The flow is therefore geodetic in general. The expressions for expansion θ, rotation ω_{ij} and shear tensor σ_{ij} are given by

$$\theta = \frac{1}{(T^2 - X^2)^{1/2}} \cdot \left[1 + \frac{(k_1 + 1)^2}{(k_1^2 + 1)}\right] \qquad \text{...(31)}$$

$$\omega_{ij} = 0 \qquad \text{...(32)}$$

$$\sigma_{11} = \frac{2k_1}{3(k_1^2 + 1)} \cdot \left[\frac{T^2}{(T^2 - X^2)^{3/2}}\right] \qquad \text{...(33)}$$

$$\sigma_{22} = \frac{\{U\}^{2k_1\mu} \cdot \{V\}^{2\nu}}{(T^2 - X^2)^{1/2}} \cdot \left[\frac{2k_1^2 - k_1 - 1}{3(k_1^2 + 1)} - \mu(k_1 - 1)\right] \qquad \text{...(34)}$$

$$\sigma_{33} = \frac{\{U\}^{2\mu} \cdot \{V\}^{2k_1\nu}}{(T^2 - X^2)^{1/2}} \cdot \left[\frac{2 - k_1(k_1 + 1)}{3(k_1^2 + 1)} - \mu(k_1 - 1)\right] \qquad \text{...(35)}$$

$$\sigma_{44} = -\left[\frac{(k_1 - 1)^2}{3(k_1^2 + 1)}\right]\left[\frac{X^2}{(T^2 - X^2)^{3/2}}\right] \qquad \text{...(36)}$$

and

$$\sigma_{14} = \left[\frac{(k_1 - 1)^2}{3(k_1^2 + 1)}\right]\left[\frac{TX}{(T^2 - X^2)^{3/2}}\right]. \qquad \text{...(37)}$$

The model is therefore expanding, non-rotating, shearing and geodetic in general. The non-vanishing components of conformal curvature tensor are

$$C_{(1212)} = \tfrac{1}{8}\left[\frac{\mu(k_1-1)\{\mu(k_1+1)-1\}}{U^2} - \frac{\nu(k_1-1)\{\nu(k_1+1)-1\}}{V^2}\right] - \frac{1}{12}\left[\frac{\mu\nu(k_1-1)^2}{UV}\right] \qquad \ldots(38)$$

$$C_{(1313)} = \tfrac{1}{8}\left[\frac{\nu(k_1-1)\{\nu(k_1+1)-1\}}{V^2} - \frac{\mu(k_1-1)\,\mu(k_1+1)-1\}}{U^2}\right] - \frac{1}{12}\left[\frac{\mu\nu(k_1-1)^2}{UV}\right] \qquad \ldots(39)$$

$$C_{(2323)} = \tfrac{1}{6}\left[\frac{\mu\nu(k_1-1)^2}{UV}\right] \qquad \ldots(40)$$

$$C_{(2424)} = \tfrac{1}{8}\left[\frac{\mu(k_1-1)\{\mu(k_1+1)-1\}}{U^2} - \frac{\nu(k_1-1)\{\nu(k_1+1)-1\}}{V^2}\right] + \frac{1}{12}\left[\frac{\mu\nu(k_1-1)^2}{UV}\right] \qquad \ldots(41)$$

$$C_{(3434)} = \tfrac{1}{8}\left[\frac{\nu(k_1-1)\{\nu(k_1+1)-1\}}{V^2} - \frac{\mu(k_1-1)\{\mu(k_1+1)-1\}}{U^2}\right] + \frac{1}{12}\left[\frac{\mu\nu(k_1-1)^2}{UV}\right] \qquad \ldots(42)$$

$$C_{(1414)} = -\tfrac{1}{6}\left[\frac{\mu\nu(k_1-1)^2}{UV}\right] \qquad \ldots(43)$$

$$C_{(1224)} = -\tfrac{1}{8}\left[\frac{\mu(k_1-1)\{\mu(k_1+1)-1\}}{U^2} + \frac{\nu(k_1-1)\{\nu(k_1-1)-1\}}{V^2}\right] \qquad \ldots(44)$$

$$C_{(1334)} = \tfrac{1}{8}\left[\frac{\mu(k_1-1)\{\mu(k_1+1)-1\}}{U^2} + \frac{\nu(k_1-1)\{\nu(k_1+1)-1\}}{V^2}\right]. \qquad \ldots(45)$$

From (38) to (45) it is clear that if $k_1 = 1$ the space-time is conformally flat. The space-time in general is of Petrov-type I. However if $k_1 \neq 1$ and

either $$\mu = \frac{1}{k_1+1} \qquad \ldots(46)$$

or $$\nu = \frac{1}{k_1+1} \qquad \ldots(47)$$

then the space-time will be of Petrov-type II. For this case the reality conditions $p > 0$ and $\rho > p$ imply that

$$\left[\frac{8k_1^2}{(k_1+1)^2(k_1^2+1)(T^2-X^2)}\right] < \Delta < \left[\frac{4k_1^2}{(k_1^2+1)(T^2-X^2)}\right]. \qquad \ldots(48)$$

For $\mu = 1/(k_1 + 1)$, we have $\nu = 2k_1/(k_1 + 1)(k_1^2 + 1)$ and the metric (23) takes the form

$$ds^2 = dT^2 - dX^2 - \{U\}^{2k_1/(k_1+1)}\{V\}^{2k_1/(k_1+1)(k_1^2+1)}\, dY^2 - \{U\}^{2/(k_1+1)}\{V\}^{4k_1^2/(k_1+1)(k_1^2+1)}\, dZ^2. \quad ...(49)$$

The physical components of C_{hijk} take the form

$$C_{AB} = \begin{bmatrix} -2\xi & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi - \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi + \eta & 0 & \eta & 0 \\ 0 & 0 & 0 & 2\xi & 0 & 0 \\ 0 & 0 & \eta & 0 & -\xi + \eta & 0 \\ 0 & \eta & 0 & 0 & 0 & -\xi - \eta \end{bmatrix}. \quad ...(50)$$

where

$$\xi = -\tfrac{1}{3}\left[\frac{(k_1 - 1)^2}{(k_1 + 1)^2 (k_1^2 + 1)}\right]\frac{1}{UV} \quad ...(51)$$

and

$$\eta = \tfrac{1}{4}\left[\frac{k_1^3 (k_1 - 1)}{(k_1 + 1)(k_1^2 + 1)^2}\right]\frac{1}{V^2}. \quad ...(52)$$

From (50) we conclude that the model exhibits gravitational radiation. For large value of X it gives us a type-two null space-time representing an outgoing radiation field, although it will not satisfy the reality conditions at $X = \infty$. The radiative term in C_{AB} is the quantity

$$\eta = \tfrac{1}{4}\left[\frac{k_1^3 (k_1 - 1)}{(k_1 + 1)(k_1^2 + 1)^2}\right]\frac{1}{V^2}.$$

REFERENCES

Ellis, G. F. R. (1971). General Relativity and Gravitation, edited by R. K. Sachs. Academic Press, New York, p. 129.

Heckmann, O., and Schucking, E. (1962). Gravitation, edited by L. Witten. Academic Press, New York, p. 438.

Jacobs, K. C. (1968). Spatially homogeneous and Euclidean cosmological models with shear. *Astrophys. J.*, **153**, 661.

———— (1969). Cosmologies of Bianchi type I with a uniform magnetic field. *Astrophys. J.*, **155**, 379.

Roy, S. R., and Singh, P. N. (1976). A cylindrically symmetric expanding universe. *J. Phys. A. Math. Nucl.*, **9**, No. 2, 255.

Roy, S. R., and Prakash, S. (1979). An anisotropic incoherent fluid cosmological model in general relativity. *Indian J. pure appl. Math.*, **10**, 1–5.

Roy, S. R., and Shyam Narain (1979). Some plane symmetric models of perfect fluid distribution in general relativity. *Indian J. pure appl. Math.*, **10**, p. 763.

Singh, K. P., and Abdussattar (1973). Plane symmetric cosmological model II. *J. Phys. A. Math. Nucl.*, **6**, 1090.

Singh, K. P., and Singh, D. N. (1968). A plane symmetric cosmological model. *Mon. Not. R. Astro. Soc.*, **140**, 453.

———— (1969). Petrov classification of cylindrically symmetric gravitational field. *J, Phys. A. Math. Nucl.* **2**, 28.

Taub, A. H. (1956). Isentropic hydrodynamics in plane symmetric space-times. *Phys. Rev.*, **103**, 454.

Thorne, K. S. (1967). Primordial element formation, primordial magnetic fields, and the isotropy of the universe. *Astrophys. J.*, **148**, 51.