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AN ASSIGNMENT PROBLEM INVOLVING ANY NUMBER OF MACHINES WITH OBJECTIVES TO MINIMIZE TOTAL PROCESSING COST AND DURATION OF COMPLETING WORK

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An assignment problem involving any number of machines with two objectives—one primary and another secondary—has been studied. The primary objective is to minimize the total processing cost and the secondary objective is to minimize the duration of completing work subject to the constraints of the problem. A two-phase algorithm has been developed to obtain the solution of this problem.

1. INTRODUCTION

In recent times, linear programming problems with more than one objective have attracted widespread interest, although such problems have been posed and solved in the past also. In the period from 1958 to 1968, linear programming problems with more than one objective have been dealt by many workers [Klahr (1958), Charnes and Cooper (1961), Zadeh (1963), Klinger (1964), DaCunha and Polak (1967), Geoffrion (1967, 1968)]. But for everything there is a season, it is only in the current decade that they have attracted widespread interest. The work of Zeleny (1974) is a classic in the field of linear multiobjective programming.

Hadley (1962) has considered a machine-assignment problem with a single objective to minimize the total processing cost. The present paper deals with the same machine-assignment problem with two objectives—one primary and another secondary. The primary objective is to minimize the total processing cost and the secondary objective is to minimize the duration of completing work subject to the constraints of the problem. A two-phase algorithm has been developed to obtain the solution of this problem. The general idea of the two-phase method is as follows: the problem is solved in two parts. First, an optimal basic feasible solution of the related machine-assignment problem which seeks to minimize the total processing cost while maximizing the total spare time available on the machines, is obtained. Then the related problem is perturbed and the solution of this perturbed problem is sought. The desired solution is obtained after solving successive perturbed problems. The method has been illustrated toward the end.

2. Formulation of the Problem

Consider a shop with m machines of the same general type. Each machine can turn out n different products and none of the products need to be processed by more than one machine. However, their suitability for turning out different products varies because of difference in their age, size, and some other characteristics. This means that the cost of operations and the processing time on one unit of the same product will vary and will depend on which machine it is processed. Let d_{ij} be the units of time required to process one unit of product j on machine i, a_i the units of time available on machine i, b_j the number of units of product j which must be completed, c_{ij} the units of cost of processing one unit of product j on machine i, and x_{ij} the number of units of product j processed on machine i in the coming time period. It is required to determine the assignment of work to the various machines fulfilling two objectives—one primary and another secondary. The primary objective is to minimize the total processing cost and the secondary objective is to minimize the duration of completing work subject to the constraints of the problem. The mathematical formulation of the problem is as follows. Find $x_{ij} \geqslant 0$ which minimize

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \ldots(1)$$

and the duration of completing work with priorities being accorded to the two objectives in the order specified above subject to the constraints

$$\sum_{j=1}^{n} d_{ij} x_{ij} \leqslant a_i \qquad (i = 1, \ldots, m) \qquad \ldots(2)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1, \ldots, n). \qquad \ldots(3)$$

3. Solution Procedure

Now we shall present a two-phase technique to obtain the solution of the problem formulated above. The notation of Hadley (1962) is followed. We introduce slack variables $x_{i(n+1)} \geqslant 0$ $(i = 1, \ldots, m)$ into the constraints (2) to transform them from inequalities to equations so as to assume the form

$$\sum_{j=1}^{n} d_{ij} x_{ij} + x_{i(n+1)} = a_i \qquad (i = 1, \ldots, m). \qquad \ldots(4)$$

We associate a cost $-\delta$ with each of the slack variables where δ is a positive number approaching zero. After doing this, the original problem is reduced to the equivalent problem which requires finding $x_{ij} \geqslant 0$ that minimize

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - \delta \sum_{i=1}^{m} x_{i(n+1)} \qquad \ldots(5)$$

and the duration of completing work with priorities being accorded to the two objectives in the order specified above subject to the constraints (4) and (3).

The duration of completing work will be minimum if the time of operation on the respective machines is equal or as close to equal as possible while the total spare time available on them is maximum possible. This observation suggests the following procedure to obtain the desired solution.

First, we consider the related problem which requires finding $x_{ij} \geqslant 0$ that minimize z given by (5) subject to the constraints (4) and (3). An optimal basic feasible solution for this related problem is obtained in the same way as for the generalized transportation problem dealt by Hadley (1962). When z given by (5) subject to the constraints (4) and (3) has attained the minimum value, the total spare time available on the machines, namely $\sum_{i=1}^{m} x_{i(n+1)}$, would have attained the maximum value as δ is a positive number approaching zero. In view of this, the optimal basic feasible solution of the related problem after setting $\delta = 0$, would yield a basic feasible solution to the original problem for which the total processing cost, namely z given by (1), is minimum while the total spare time available on the machines, namely $\sum_{i=1}^{m} x_{i(n+1)}$, is maximum. We denote by z_0 the value of z for this optimal basic feasible solution of the related problem after setting $\delta = 0$. We then determine the time of operation on the respective machines for this solution. Suppose that we find that the time of operation is maximum on machine s among all the machines. Then we try to reduce the time of operation on machine s so that the time of operation on the respective machines is equal or as close to equal as possible with the total processing cost remaining minimum at z_0. If the time of operation on machine s cannot be reduced in the specified manner, then the time of operation on machine s is the shortest duration of completing work and the optimal basic feasible solution of the related problem is the desired solution to the original problem. If we are able to reduce the time of operation on machine s in the specified manner but find that it still remains greater than that on the remaining machines, the procedure is repeated. This repetition is continued till we reach the state when the time of operation on machine s is either equal to that on the remaining machines or is greater than that on the remaining machines but cannot be further reduced in the specified manner. Then this time of operation on machine s is the shortest duration of completing work and we have attained the desired solution to the original problem. If however the reduced time of operation on machine s turns out to be less than that on one or more of the remaining machines, we select the machine on which the time of operation is maximum among all the machines and try to reduce the time of operation on it in the way described for that on machine s. This process is continued till we reach the state when the time of operation on the selected machine is either equal to that on the remaining machines or is greater than that on the remaining machines but cannot be further reduced in the specified manner. And when this happens, we have attained the desired solution to the original problem.

Below we present a procedure to reduce the time of operation on machine s so that the time of operation on the respective machines is equal or as close to equal as possible with the total processing cost remaining minimum at z_0. We denote the spare time available on machine s by b_{n+1}. We then perturb b_{n+1} to the value $(b_{n+1} + \epsilon)$, where ϵ is a nonnegative number. The constraint which seeks that $(b_{n+1} + \epsilon)$ units of spare time be available on machine s after processing the requisite number of products may be mathematically expressed as follows:

$$a_s - \sum_{j=1}^{n} d_{sj}x_{sj} = b_{n+1} + \epsilon. \qquad \text{...(6)}$$

We then consider the following perturbed problem whose solution would yield a solution which would reduce the time of operation on machine s so that the time of operation on the respective machines is equal or as close to equal as possible with the total processing cost remaining minimum at z_0. Find $x_{ij} \geqslant 0$ which minimize z given by (1) to the value z_0, while assigning a nonnegative value to ϵ so that the time of operation on the respective machines is equal or as close to equal as possible, subject to the constraints (4), (3), (6). It is worth noting that in the perturbed problem, we associate zero costs with the slack variables. If the value of ϵ turns out to be zero, then it means that it is not possible to reduce the time of operation on machine s in the specified manner and the optimal basic feasible solution of the related problem is the desired solution to the original problem. However, if ϵ turns out to be positive, say $\epsilon = \epsilon_1$, then we have a solution for which the time of operation on machine s has been reduced to $(a_s - b_{n+1} - \epsilon_1)$. To further reduce the time of operation on machine s in the specified manner, we proceed as follows. We replace b_{n+1} by $(b_{n+1} + \epsilon_1)$ in eqn. (6) which then assumes the form

$$a_s - \sum_{j=1}^{n} d_{sj}x_{sj} = b_{n+1} + \epsilon_1 + \epsilon. \qquad \text{...(7)}$$

We then seek to find $x_{ij} \geqslant 0$ which minimize z given by (1) to the value z_0, while assigning a nonnegative value to ϵ so that the time of operation on the respective machines is equal or as close to equal as possible, subject to the constraints (4), (3), (7). This process is continued till we reach the state when the time of operation on machine s is either equal to that on the respective machines or is less than that on one or more of the machines or is greater than that on the remaining machines but cannot be further reduced in the specified manner.

4. First Perturbed Problem

Below we give the details of the procedure to solve the first perturbed problem involved in the process of finding the desired solution to the original problem. Here we determine $x_{ij} \geqslant 0$ that minimize z given by (1) to the value z_0, while assigning a nonnegative value to ϵ so that the time of operation on the respective machines is equal or as close to equal as possible, subject to the constraints (4), (3), (6). In this perturbed problem, we associate zero costs with the slack variables as stated earlier.

By the aid of eqns. (4), eqn. (6) reduces to the form

$$x_{s(n+1)} = b_{n+1} + \epsilon. \quad \text{...(8)}$$

To overcome the difficulty in obtaining an initial basic feasible solution of the proposed problem, we introduce an artificial variable $x_{(m+1)(n+1)} \geqslant 0$ into eqn. (8) which then assumes the form

$$x_{s(n+1)} + x_{(m+1)(n+1)} = b_{n+1} + \epsilon. \quad \text{...(9)}$$

We associate a cost M with the artificial variable where M is an arbitrarily large positive number.

After this, the problem of interest reduces to the following equivalent problem. Find $x_{ij} \geqslant 0$ which minimize

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + Mx_{(m+1)(n+1)} \quad \text{...(10)}$$

to the value z_0, while assigning a nonnegative value to ϵ so that the time of operation on the respective machines is equal or as close to equal as possible, subject to the constraints (4), (3), (9). It may be pointed out that the minimum value of z given by (10) cannot be less than z_0 because there are now more restrictions on the ways of assigning the work to the machines. The tableau representation of this problem is shown in Table I. In Table I, row i of the tableau is denoted by R_i and its column j is denoted by P_j. The first m cells in P_{n+1} correspond to the slack variables $x_{i(n+1)}$ $(i = 1, ..., m)$ and the $(m + 1)$th cell in it corresponds to the artificial variable $x_{(m+1)(n+1)}$. All other cells correspond to the legitimate variables $x_{ij}(i = 1, ..., m;$ $j = 1, ..., n)$. All the cells in P_{n+1} except the sth and $(m + 1)$th cells in it have an entry of zero in their right bottom corners. The sth and $(m + 1)$th cells in P_{n+1} have an entry of unity in their right bottom corners. The first n cells in R_{m+1} are blank whereas there is no entry in the left bottom corner of the $(n + 1)$th cell in this row. The left top corner of all the cells (i, j) contains the cost of processing one unit of product j on machine i. To obtain a_i in $R_i(i = 1, ..., m)$, sum the products obtained multiplying x_{ij} by the entry in the left bottom corner of the associated cell (i, j) across the row. To obtain b_j in $P_j(j = 1, ..., n + 1)$, sum the products obtained multiplying x_{ij} by the entry in the right bottom corner of the associated cell (i, j) across the column. And z is obtained by summing the products obtained multiplying x_{ij} by the entry in the left top corner of the associated cell (i, j) all over the tableau.

TABLE I

	P_1		P_2			P_n		P_{n+1}		a_i	u_i
R_1	c_{11}		c_{12}		...	c_{1n}		0		a_1	u_1
	d_{11}	1	d_{12}	1		d_{1n}	1	1	0		
R_2	c_{21}		c_{22}		...	c_{2n}		0		a_2	u_2
	d_{21}	1	d_{22}	1		d_{2n}	1	1	0		
	$\vdots$		$\vdots$		...	$\vdots$		$\vdots$		$\vdots$	$\vdots$
R_m	c_{m1}		c_{m2}		...	c_{mn}		0		a_m	u_m
	d_{m1}	1	d_{m2}	1		d_{mn}	1	1	0		
R_{m+1}					...			M			
									1		
b_j	b_1		b_2		...	b_n		$b_{n+1} + \epsilon$			
v_j	v_1		v_2		...	v_n		v_{n+1}			

An initial basic feasible solution for this problem is provided by the optimal basic feasible solution of the related problem which requires finding $x_{ij} \geqslant 0$ that minimize z given by (5) subject to the constraints (4) and (3), and $x_{(m+1)(n+1)} = b_{n+1} + \epsilon$. The values of the basic variables of this initial solution are entered in the right top corners of the associated cells and are bracketed so that the basic cells are distinguished from the other cells.

To determine whether the basic feasible solution is optimal, we require the values of the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells (i, j). To compute the $(z_{ij} - c_{ij})$, we proceed as follows. Denote by $c^B_{\alpha\beta}$ the costs corresponding to the basic variables associated with the basic cells (α, β). Then we compute the m $u_i (i = 1, \ldots, m)$ and $(n + 1)$ $v_j (j = 1, \ldots, n + 1)$ from the following $(m + n + 1)$ equations of four different forms

$$\left.\begin{aligned} d_{\alpha\beta} u_\alpha + v_\beta &= c^B_{\alpha\beta} \quad \begin{pmatrix}\text{for the basic cells corresponding to the} \\ \text{legitimate variables}\end{pmatrix} \\ u_\alpha &= 0 \quad \begin{pmatrix}\text{for the basic cells corresponding to the} \\ \text{slack variables in the rows other than } R_s\end{pmatrix} \\ u_\alpha + v_\beta &= 0 \quad \begin{pmatrix}\text{for the basic cell corresponding to the} \\ \text{slack variable in the row } R_s\end{pmatrix} \\ v_\beta &= M \quad \begin{pmatrix}\text{for the basic cell corresponding to the} \\ \text{artificial variable}\end{pmatrix}. \end{aligned}\right\} \quad \ldots(11)$$

Once the m u_i and $(n + 1)$ v_j are known, we can compute the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells from the following formulas

$$\left.\begin{aligned}
z_{ij} - c_{ij} &= d_{ij}u_i + v_j - c_{ij} && \left(\begin{array}{l}\text{for the nonbasic cells corresponding to the}\\ \text{legitimate variables}\end{array}\right)\\
z_{ij} - c_{ij} &= u_i && \left(\begin{array}{l}\text{for the nonbasic cells corresponding to the}\\ \text{slack variables in the rows other than } R_s\end{array}\right)\\
z_{ij} - c_{ij} &= u_i + v_j && \left(\begin{array}{l}\text{for the nonbasic cell corresponding to the}\\ \text{slack variable in the row } R_s\end{array}\right)\\
z_{ij} - c_{ij} &= v_j - M && \left(\begin{array}{l}\text{for the nonbasic cell corresponding to the}\\ \text{artificial variable}\end{array}\right).
\end{aligned}\right\} \quad \ldots(12)$$

After the values of the $(z_{ij} - c_{ij})$ have been calculated, they are entered in the right top corners of the nonbasic cells (i, j). If all the $(z_{ij} - c_{ij})$ are nonpositive, then the basic feasible solution is optimal.

On the other hand, if one or more of the $(z_{ij} - c_{ij})$ is or are positive, the basic feasible solution is not optimal. Then the vector $\bar{p}_{st}$ corresponding to the cell (s, t) to enter the basis is determined from $(z_{st} - c_{st}) = \max (z_{ij} - c_{ij})$ among all the positive $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells. To determine the vector to leave the basis, the $y_{st}^{\alpha\beta}$ are computed. They are determined from the set of $(m + n + 1)$ equations

$$\bar{p}_{st} = \sum_{\alpha\beta} y_{st}^{\alpha\beta} \bar{p}_{\alpha\beta}^{B} \quad \ldots(13)$$

where $\sum_{\alpha\beta}$ means the summation over the basis vectors. After having calculated the $y_{st}^{\alpha\beta}$, the criterion used in the simplex method is employed to determine the vector $\bar{p}_{qr}^{B}$ to leave the basis, i.e.,

$$\frac{x_{qr}^{B}}{y_{st}^{qr}} = \theta = \min \left\{\frac{x_{\alpha\beta}^{B}}{y_{st}^{\alpha\beta}}\right\}, \; y_{st}^{\alpha\beta} > 0 \quad \ldots(14)$$

The determination of this θ imposes certain restrictions on ϵ. The new basic feasible solution is obtained by setting $\hat{x}_{st} = \theta$, $\hat{x}_{qr} = 0$ and making the appropriate adjustments in the tableau to obtain the values of the remaining basic variables.

For this new basic feasible solution, a new tableau is constructed and the whole previous procedure is repeated. This repetition is continued till we reach the state where all the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells are nonpositive. And when this happens, we have attained an optimal basic feasible solution of the perturbed problem; however the values of the basic variables in this solution contain ϵ. The value of ϵ, say $\epsilon = \epsilon_1$, is so chosen that the time of operation on the respective machines is equal or as close to equal as possible while satisfying the restrictions imposed on it in the course of the procedure of determining the optimal solution and also the restriction which requires the optimal values of z given by (10) to be equal to z_0. The optimal basic feasible solution of this perturbed problem when this value of ϵ has been substituted would yield a solution for which the time of operation on machine s is $(a_s - b_{n+1} - \epsilon_1)$.

5. Two Numerical Examples

Now we shall apply the above procedure to obtain the solution of two numerical problems which result by taking $n = 5$, $m = 2$ and assigning numerical values to all other quantities in the problem formulated above in section 2. The tableau representation of the related problem associated with the first numerical problem is shown in Table II. It should be noted that the first, second, third, fourth, fifth cells in R_1 and R_2 correspond to the legitimate variables $x_{ij}(i = 1, 2;\ j = 1, 2, 3, 4, 5)$, and the sixth cell each in R_1 and R_2 corresponds to a slack variable. For this related problem, the objective function which we seek to minimize is

$$z = x_{11} + 2x_{12} + 2x_{13} + 3x_{14} + 3x_{15} + 2x_{21} + x_{22}$$
$$2x_{23} + 2x_{24} + 3x_{25} - \delta(x_{16} + x_{26}). \qquad \ldots(15)$$

To find an initial basic feasible solution for this related problem, we apply the column-minima method after some modification. In the case of the standard transportation problem, x_{ij} is the amount used of resource a_i; it is also the amount satisfied of requirement b_j. But in the case of the present problem, x_{ij} multiplied by the entry in the left bottom corner of the associated cell (i, j) is the amount used of resource a_i and x_{ij} multiplied by the entry in the right bottom corner of the associated cell (i, j) is the amount satisfied of requirement b_j. Following the procedure, an initial basic feasible solution is obtained and the values of the basic variables of this solution are entered inside brackets in the right top corners of the associated cells in the tableau of Table II.

To determine whether the basic feasible solution is optimal, we require the values of the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells. For this purpose, we first calculate the values of the u_i and v_j, and enter their values in the last column and last row of the tableau of Table II. Then the values of the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells are calculated and are entered in the right top corners of the associated nonbasic cells in the tableau of Table II. We observe that all the $(z_{ij} - c_{ij})$ are nonpositive; this means that the basic feasible solution is optimal. This optimal basic feasible solution of the related problem, after setting $\delta = 0$, yields a basic feasible solution to the first numerical problem for which the total processing cost is minimum while the total spare time available on the machines is maximum. The total processing cost for this optimal basic feasible solution of the related problem after setting $\delta = 0$ comes out to be

$$z_0 = 1 \times 200 + 2 \times 300 + 3 \times 100 + 1 \times 500 + 2 \times 100$$
$$= 1800. \qquad \ldots(16)$$

The time of operation on machine 1 and that on machine 2 is 1800 units and 1500 units respectively for this solution, and is thus greater on machine 1. And the spare time available on machine 1 for this solution is given by

$$b_6 = 200. \qquad \ldots(17)$$

We perturb b_6 to the value $b_6 + \epsilon$.

We then consider the following perturbed problem whose solution would yield a solution which would reduce the time of operation on machine 1 so that the time of operation on the respective machines is equal or as close to equal as possible with the total processing cost remaining minimum at $z_0 = 1800$. The tableau representation of this perturbed problem is shown in Table III. For this perturbed problem, we seek to determine $x_{ij} \geqslant 0$ which minimize z given by

$$z = x_{11} + 2x_{12} + 2x_{13} + 3x_{14} + 3x_{15} + 2x_{21} + x_{22}$$
$$+ 2x_{23} + 2x_{24} + 3x_{25} + Mx_{36} \qquad \ldots(18)$$

to the value $z_0 = 1800$, while assigning a nonnegative value to ϵ so that the time of operation on the respective machines is equal or as close to equal as possible, subject to the constraints of the problem.

An initial basic feasible solution for this perturbed problem is immediately obtained with the aid of the optimal solution of the related problem. The values of the basic variables of this initial solution are entered inside brackets in the right top corners of the associated cells in the tableau of Table III.

To determine whether the basic feasible solution is optimal, we require the values of the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells. For this purpose, we first calculate the values of the u_i and v_j, and enter their values in the last column and last row of the tableau of Table III. Then the values of the $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells are calculated and are entered in the right top corners of the associated cells in the tableau of Table III. We now observe that $(z_{21} - c_{21})$, $(z_{23} - c_{23})$, $(z_{25} - c_{25})$ are positive and that the second one is greatest among them; therefore the basic feasible solution is not optimal and the vector $\bar{p}_{23}$ should enter the basis.

To determine the vector to leave the basis, the $y_{23}^{\alpha\beta}$ are computed. They are determined from the set of equations

$$\left.\begin{array}{l} 2y_{23}^{11} + 4y_{23}^{13} + 2y_{23}^{15} + y_{23}^{16} = 0,\ 2y_{23}^{22} + 5y_{23}^{24} + y_{23}^{26} + y_{23}^{36} = 4, \\ y_{23}^{11} = 0,\ y_{23}^{22} = 0,\ y_{23}^{13} = 1,\ y_{23}^{24} = 0,\ y_{23}^{15} = 0,\ y_{23}^{16} + y_{23}^{36} = 0 \end{array}\right\} \qquad \ldots(19)$$

TABLE II

	P_1		P_2		P_3		P_4		P_5		P_6		a_i	u_i
R_1	1	{200}	2	-1	2	{300}	3	$-3\delta-1$	3	{100}	$-\delta$	{200}	2000	$-\delta$
	2	1	2	1	4	1	8	1	2	1	1			
R_2	2	$-2\delta-1$	1	{500}	2	0	2	{100}	3	0	$-\delta$	{100}	1600	$-\delta$
	4	1	2	1	4	1	5	1	2	1	1			
b_j	200		500		300		100		100					
v_j	$2\delta+1$		$2\delta+1$		$4\delta+2$		$5\delta+2$		$2\delta+3$					

TABLE III

	P_1		P_2		P_3		P_4		P_5		P_6		a_i	u_i
R_1	1	{200}	2	$-2M-1$	2	{300}	3	$-8M-1$	3	{100}	0	{200}	2000	$-M$
	2	1	2	1	4	1	8	1	2	1	1	1		
R_2	2	$2M-1$	1	{500}	2	$4M$	2	{100}	3	$2M$	0	{100}	1600	0
	4	1	2	1	4	1	5	1	2	1	1	0		
R_3											M	$\{\epsilon\}$		
												1		
b_j	200		500		300		100		100		$200+\epsilon$			
v_j	$2M+1$		1		$4M+2$		2		$2M+3$		M			

These equations at once yield the solution

$$y_{23}^{11} = 0, y_{23}^{13} = 1, y_{23}^{15} = 0, y_{23}^{16} = -4, y_{23}^{22} = 0, y_{23}^{24} = 0$$
$$y_{23}^{26} = 0, y_{23}^{36} = 4.$$

Only y_{23}^{13} and y_{23}^{36} are positive and

$$\min \left\{ \frac{x_{13}}{y_{23}^{13}}, \frac{x_{36}}{y_{23}^{36}} \right\} = \min \left\{ 300, \frac{\epsilon}{4} \right\} = \frac{\epsilon}{4}$$

if $\epsilon \leqslant 1200$.

Therefore the vector $\bar{p}_{36}$ should leave the basis. The new basic feasible solution is obtained by setting $\hat{x}_{23} = \epsilon/4$, $\hat{x}_{36} = 0$ and making the appropriate adjustments in the tableau of Table III to obtain the values of the remaining basic variables. The values of the new basic variables are entered inside brackets in the tableau of Table IV.

The values of the u_i, v_j, $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells are calculated in the same way as done earlier, and are entered in the appropriate places in the tableau of Table IV. We observe that all the $(z_{ij} - c_{ij})$ are now nonpositive; this means that the new basic feasible solution is optimal. However, the values of the basic variables in this optimal solution contain ϵ. The value of z for this optimal solution is given by

$$\begin{aligned} z &= 1 \times 200 + 2 \times (300 - \tfrac{1}{4}\epsilon) + 3 \times 100 + 1 \times 500 + 2 \times \tfrac{1}{4}\epsilon \\ &\quad + 2 \times 100 \\ &= 1800 \end{aligned} \qquad \ldots(20)$$

which incidentally does not contain ϵ. The value of ϵ is to be so chosen that the time of operation on the respective machines is equal or as close to equal as possible while satisfying the conditions that $0 \leqslant \epsilon \leqslant 1200$ and z given by (20) is equal to $z_0 = 1800$. So the acceptable value of ϵ is 100. This means that the optimal basic feasible solution of the first perturbed problem after substituting $\epsilon = 100$ yields a solution reducing the duration of completing work from 1800 units to 1700 units with the total processing cost remaining minimum at $z_0 = 1800$. Though the time of operation on machine 1 has been reduced in the specified manner, it is still greater than that on machine 2. So we consider the second perturbed problem to further reduce the time of operation on machine 1 so that the time of operation on the respective machines is equal or as close to equal as possible with the total processing cost remaining minimum at $z_0 = 1800$. The tableau representation of this second perturbed problem is shown in Table V. For this perturbed problem, we anew seek to find $x_{ij} \geqslant 0$ which minimize z given by (18) to the value $z_0 = 1800$, while assigning a nonnegative value to ϵ so that the time of operation on the respective machines is equal or as close to equal as possible, subject to the constraints of the problem.

TABLE IV

	P_1		P_2		P_3		P_4		P_5		P_6		a_i	u_i
R_1	1	{200}	2	-1	2	$\{300-\frac{1}{4}\epsilon\}$	3	-1	3	{100}	0	$\{200+\epsilon\}$	2000	0
	2	1	2	1	4	1	8	1	2	1	1	1		
R_2	2	-1	1	{500}	2	$\{\frac{1}{4}\epsilon\}$	2	{100}	3	0	0	$\{100-\epsilon\}$	1600	0
	4	1	2	1	4	1	5	1	2	1	1	0		
R_3											M	$-M$		
												1		
b_j	200		500		300		100		100		$200+\epsilon$			
v_j	1		1		2		2		3		0			

TABLE V

	P_1		P_2		P_3		P_4		P_5		P_6		a_i	u_i
R_1	1	{200}	2	-1	2	{275}	3	$-3M-1$	3	{100}	0	{300}	2000	$-M$
	2	1	2	1	4	1	8	1	2	1	1	1		
R_2	2	$-2M-1$	1	{500}	2	{25}	2	{100}	3	0	0	$-M$	1600	$-M$
	4	1	2	1	4	1	5	1	2	1	1	0		
R_3											M	$\{\epsilon\}$		
												1		
b_j	200		500		300		100		100		$300+\epsilon$			
v_j	$2M+1$		$2M+1$		$4M+2$		$5M+2$		$2M+3$		M			

An initial basic feasible solution for this second perturbed problem is immediately obtained with the aid of the optimal solution of the first perturbed problem. The values of the basic variables of the initial solution for this problem are entered inside brackets in the right top corners of the associated cells in the tableau of Table V. The values of the u_i, v_j, $(z_{ij} - c_{ij})$ corresponding to the nonbasic cells are calculated in the same way as done earlier, and are entered in the appropriate places in the tableau of Table V. We observe that all the $(z_{ij} - c_{ij})$ are nonpositive and therefore the basic feasible solution is optimal. However, the value of one of the basic variables in this optimal solution contains ϵ. The value of z for this optimal solution is given by

$$\begin{aligned} z &= 1 \times 200 + 2 \times 275 + 3 \times 100 + 1 \times 500 + 2 \times 25 \\ &\quad + 2 \times 100 + M \times \epsilon \\ &= 1800 + M \times \epsilon. \end{aligned} \qquad \ldots(21)$$

The value of ϵ again is to be so chosen that the time of operation on the respective machines is equal or as close to equal as possible while satisfying the conditions that $\epsilon \geqslant 0$ and z given by (21) is equal to $z_0 = 1800$. So the acceptable value of ϵ is zero. This means that it is not possible to further reduce the time of operation on machine 1 in the specified manner, and therefore the optimal solution of the first perturbed problem after substituting $\epsilon = 100$ yields the desired solution to the first numerical problem minimizing the total processing cost and the duration of completing work to 1800 units and 1700 units respectively.

Next, we consider a variant problem obtained by taking 1700 instead of 1600 for a_2 in the first numerical problem discussed above to illustrate the case where the time of operation on the respective machines can be made equal for the value of ϵ chosen in the specified manner. The solution procedure for the variant problem is exactly the same as for the first numerical problem but terminates with the solution of the first perturbed problem. The optimal basic feasible solution of the first perturbed problem associated with the variant problem is provided by the tableau of Table IV, after replacing 1600 by 1700 for a_2 and $(100 - \epsilon)$ by $(200 - \epsilon)$ in the right top corner of the cell (2, 6) in it. The acceptable value of ϵ for this problem is clearly 150 and the time of operation on the respective machines for this value of ϵ is equal and is 1650 units. Therefore the modified tableau of Table IV after substituting $\epsilon = 150$ yields the desired solution to the variant problem minimizing the total processing cost and the duration of completing work to 1800 units and 1650 units respectively.

Thus, of the two numerical problems considered above out of which one is the variant of the other, the minimum total processing cost is the same for both the problems but the minimum duration of completing work is greater for the original problem than for its variant one. This is understandable in view of the fact that there is less flexibility in assigning the work to the two machines in the original problem compared to its variant one because less time is available on machine 2 in the original problem and equal time is available on machine 1 in both the problems.

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