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COMMON FIXED POINT THEOREMS FOR A FAMILY OF MAPPINGS

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A recent theorem of Yeh (1979) concerning common fixed points for a family of self-mappings of a complete metric space has been improved. Convergence theorems for sequences of mappings and their common fixed points are also studied. The present work generalizes some previously known results.

1. INTRODUCTION

Recently, Yeh (1979) proved an interesting extension (Theorem A below) of a common fixed point theorem due to Jungck (1976). Throughout the paper H denotes a family of mappings such that each $h \in H$, $h : (R^+)^5 \to R^+$, and h is upper semicontinuous and nondecreasing in each coordinate variable. R^+ stands for the set of nonnegative reals.

Theorem A — Let E, F and T be three continuous self-mappings of a complete metric space (X, d) satisfying conditions:

(C_1) $ET = TE$, $FT = TF$, $E(X) \subset T(X)$ and $F(X) \subset T(X)$;

(C_2) there exists an h in H such that for all x, y in X, $d(Ex, Fy) \leqslant h(d(Tx, Ty), d(Tx, Ex), d(Tx, Fy), d(Ty, Ex), d(Ty, Fy))$ where h satisfies the condition :

(C_3) $g(t) \equiv h(t, t, at, bt, t) < t$ for each t in $R^+ - \{0\}$, where $a + b = 2$. Then E, F and T have a unique common fixed point in X.

(*Note* : From the proof of Theorem A in Yeh (1979), it appears that in (C_3), $a + b$ should be equal to 3).

In this note we show that the continuity condition of the mappings E and F in Theorem A are unnecessary. We also discuss the convergence of sequences of mappings and their common fixed points when the mappings involved satisfy condition (C_2). It may be remarked that in the absence of continuity hypothesis of mappings E and F, our results are indeed extensions of those obtained by Husain and Sehgal (1975), Iséki (1974), Singh and Meade (1977), Srivastava and Gupta (1971), Reich (1971), Sehgal (1972, 1974) and others.

2. Main Results

Theorem 1 — Let E, F and T be three self-mappings of a complete metric space (X, d) satisfying conditions (C_1) and (C_2) of Theorem A. Further, let T be continuous. Then E, F and T have a unique common fixed point in X.

Proof : Let x_0 be any point in X. Let $x_1 \in X$ be such that $Tx_1 = Ex_0$, and $x_2 \in X$ such that $Tx_2 = Fx_1$. In general, one can choose x_{2n+1} and x_{2n+2} such that

$$Tx_{2n+1} = Ex_{2n}, \quad Tx_{2n+2} = Fx_{2n+1} \qquad \ldots(*)$$

for $n = 0, 1, \ldots$.

Then, as in the proof of Theorem 1 of Yeh (1979), we can prove that $\{Tx_n\}$ is a Cauchy sequence. By the completeness of X, $\{Tx_n\}$ converges to a point $z \in X$. It follows from (*) that $\{Ex_{2n}\}$ and $\{Fx_{2n+1}\}$ also converges to z. Using continuity of T and (C_1), we see that $TTx_n \to Tz$, $ETx_{2n} \to Tz$ and $FTx_{2n+1} \to Tz$. Now we show that $ETx_{2n} \to Ez$. To do this consider the inequality

$$d(ETx_{2n}, Ez) \leqslant d(ETx_{2n}, FTx_{2n+1}) + d(FTx_{2n+1}, Ez).$$

From (C_2) we have

$$d(Ez, FTx_{2n+1}) \leqslant h(d(Tz, TTx_{2n+1}), d(Tz, Ez), d(Tz, FTx_{2n+1}), d(TTx_{2n+1}, Ez), d(TTx_{2n+1}, FTx_{2n+1})).$$

We shall make use of the following inequalities

$$d(Tz, Ez) \leqslant d(Tz, ETx_{2n}) + d(ETx_{2n}, Ez)$$

and

$$d(TTx_{2n+1}, Ez) \leqslant d(TTx_{2n+1}, Tz) + d(Tz, ETx_{2n}) + d(ETx_{2n}, Ez).$$

Put $t = \lim_{n\to\infty} d(Ez, ETx_{2n})$ and suppose that $t > 0$. Then letting $n \to \infty$, and using the above inequalities and upper semicontinuity of h, we have

$$t \leqslant h(0, t, 0\ t, 0) \leqslant g(t) < t,$$

a contradiction. Hence $t = 0$ implying thereby that $ETx_{2n} \to Ez$. But X is Hausdorff so $Ez = Tz$. Similarly, we can prove that $Fz = Tz$. The rest of the proof is identical with Theorem 1 of Yeh (1979). This completes the proof.

As a slight generalization of Theorem 1, we have the following result.

Theorem 2 — Let E, F and T be three self-mappings of a complete metric space (X, d) such that the following hold:

(C_1') $ET = TE$, $FT = TF$, $E(X) \subset T^r(X)$ and $F(X) \subset T^r(X)$, where r is some positive integer;

(C_2') there exist positive integers p, q, and $h \in H$ such that for all $x, y \in X$

$$d(E^p x, F^q y) \leqslant h(d(T^r x, T^r y), d(T^r x, E^p x), d(T^r x, F^q y), d(T^r y, E^p x), d(T^r y, F^q y))$$

where h satisfies (C_3) of Theorem A. If T^r is continuous, then E, F and T have a unique common fixed point.

PROOF : Since E commutes with T, E^p also commutes with T^r, and

$$E^p(X) \subset E(X) \subset T^r(X).$$

Similarly, F^q commutes with T^r and $F^q(X) \subset T^r(X)$. Thus Theorem 1 is applicable to E^p, F^q and T^r, so there exists a unique $z \in X$ such that

$$z = E^p z = F^q z = T^r z.$$

This gives

$$Tz = T^r(Tz) = E^p(Tz) = F^q(Tz).$$

Therefore Tz is a common fixed point of T^r, E^p and F^q. Also

$$Ez = T^r(Ez) = E^p(Ez).$$

So Ez is a common fixed point of T^r and E^p. Similarly, Fz is a common fixed point of T^r and F^q. Using (C_2') for $x = Ez$ and $y = Fz$ we get $Ez = Fz$. Thus $Ez = Fz = Tz$ are common fixed points of E^p, F^q and T^r. The unicity of z then implies that z is a unique common fixed point of E, F and T. This completes the proof.

Remark : Theorem 1 and Theorem 2 show that we can drop continuity requirement of certain mappings from Corollary 1, Theorem 2 and Theorem 3 of Yeh (1979).

Putting $T = I_X$, the identity mapping on X, in Theorem 1, we get the following result of Singh and Meade (1977).

Theorem B — Let (X, d) be a complete metric space and let E and F be self-mappings of X. Suppose there exists an $h \in H$ such that for all $x, y \in X$

$$d(Ex, Fy) \leqslant h(d(x, y), d(x, Ex), d(x, Fy), d(y, Ex), d(y, Fy))$$

where h satisfies the condition: for any $t > 0$

$$h(t, t, a_1 t, a_2 t, t) < t \quad \text{with} \quad a_1 + a_2 = 3.$$

Then there exists a $z \in X$ such that z is a unique common fixed point of E and F.

Recall that Theorem B is a slightly revised version of the main result of Husain and Sehgal (1975). Also note that the results like Theorem 2, Corollary 1 and Corollary 2 of Husain and Sehgal (1975) would then follow from the corresponding results derived from our Theorem 1.

We also remark that after a careful examination it is found that the Banach contraction principle and several other fixed point theorems do not require all the metric properties, particularly the axiom of triangular inequality in their proofs by the method of iteration. With this in mind Kasahara (1976a) has published a very useful treatment by introducing the notion of L-spaces. We observe that our Theorem 1, with some restriction on H, can be extended to the case when X is a L-space. As a sample, we state the following result in which H denotes a family of mappings $h : (R^+)^3 \to R^+$ and h is upper semicontinuous and nondecreasing in each coordinate variable.

Theorem 3 — Let E, F and T be continuous mappings of a separated L-space $(X, \to)$ into itself satisfying (C_1). Suppose that $(X, \to)$ is d-complete for some semi-metric d on X, and there exists an $h \in H$ such that for all $x, y \in X$

$$d(Ex, Fy) \leqslant h(d(Tx, Ty), d(Tx, Ex), d(Ty, Fy))$$

where h satisfies the condition: for $t > 0$

$$h(t, t, t) < t.$$

Then E, F and T have a unique common fixed point in X.

Remarks : (i) As in Theorem 1, we can avoid the continuity of the mappings E and F in Theorem 3 provided d is continuous.

(ii) Various results due to Iséki (1975), Kasahara (1976b, 1976c) and Singh (1979) can be viewed as particular cases of Theorem 3.

Now we wish to examine the convergence of sequences and their fixed points.

Theorem 4 — Let $\{E_n\}$, $\{F_n\}$ and $\{T_n\}$ be sequences of self-mappings of a complete metric space (X, d) such that $\{E_n\}$, $\{F_n\}$ and $\{T_n\}$ converge uniformly to self-mappings E, F and T of X with T continuous. Suppose that for each $n \geqslant 1$, x_n is a common fixed point of E_n and T_n, and y_n is a common fixed point of F_n and T_n. Further, let E, F and T satisfy conditions (C_1) and (C_2). If x_0 is the common fixed point of E, F and T, and $\sup d(x_n, x_0) < \infty$ and $\sup d(y_n, x_0) < \infty$ then $x_n \to x_0$ and $y_n \to x_0$

PROOF : Theorem 1 ensures the existence of the unique common fixed point x_0 of E, F, and T. Since $E_n \to E$ and $T_n \to T$ uniformly, it follows that

$$d(E_n x_n, E x_n) = d(x_n, E x_n) \to 0 \quad \text{and} \quad d(T_n x_n, T x_n) = d(x_n, T x_n) \to 0$$

as $n \to \infty$. Let $\epsilon = \sup d(x_n, x_0)$. Then, since

$$d(Tx_n, x_0) \leqslant d(Tx_n, x_n) + d(x_n, x_0)$$

$$d(Tx_n, Ex_n) \leqslant d(Tx_n, T_n x_n) + d(E_n x_n, Ex_n)$$

and

$$d(x_0, Ex_n) \leqslant d(x_0, x_n) + d(x_n, Ex_n)$$

it follows from (C_2) that

$$\begin{aligned} d(x_n, x_0) &\leqslant d(x_n, Ex_n) + d(Ex_n, Fx_0) \\ &\leqslant d(x_n, Ex_n) + h(d(Tx_n, Tx_0), d(Tx_n, Ex_n), \\ &\qquad d(Tx_n, Fx_0), d(Tx_0, Ex_n), d(Tx_0, Fx_0)) \\ &= d(x_n, Ex_n) + h(d(Tx_n, x_0), d(Tx_n, Ex_n), \\ &\qquad d(Tx_n, x_0), d(x_0, Ex_n), 0). \end{aligned}$$

This implies that

$$\epsilon \leqslant h(\epsilon, 0, \epsilon, \epsilon, 0) \leqslant g(\epsilon) < \epsilon$$

and hence $\epsilon = 0$ and, consequently, $x_n \to x_0$. Similarly, we can show that $y_n \to x_0$. This completes the proof.

Remark : If $E_n = F_n$ for each n and $T_n = I_x$ for each n, then Theorem 4 reduces to Theorem 3 of Husain and Sehgal (1975) which is in turn an improvement of Theorem 2 of Iséki (1974).

Our next result is an application of Theorem 4.

Theorem 5 — Let $\{E_n\}$, $\{F_n\}$ and $\{T_n\}$ be sequences of self-mappings of a complete metric space (X, d) such that T_n is continuous for each n, and E_n, F_n and T_n satisfy conditions (C_1) and (C_2) for each n. If E, F and T are uniform limits of E_n, F_n and T_n respectively, then E, F and T satisfy conditions (C_1) and (C_2). Also, the sequence $\{x_n\}$ of unique common fixed points of E_n, F_n and T_n converges to the unique common fixed point x_0 of E, F and T, whenever $\sup d(x_n, x_0) < \infty$.

Proof : Since E_n, F_n and T_n satisfy condition (C_1) and they converge uniformly to E, F, and T respectively, it follows from $E_nT_n = T_nE_n$, $F_nT_n = T_nF_n$, $E_n(X) \subset T_n(X)$ and $F_n(X) \subset T_n(X)$ that

$$ET = TE, FT = TF, E(X) \subset T(X) \quad \text{and} \quad F(X) \subset T(X).$$

Therefore E, F and T satisfy (C_1).

Now consider, for arbitrary $x, y \in X$ the following inequality.

$$d(Ex, Fy) \leqslant d(Ex, E_nx) + d(E_nx, F_ny) + d(F_ny, Fy).$$

Then from (C_2) we have

$$\begin{aligned} d(Ex, Fy) \leqslant d(Ex, E_nx) + h(d(T_nx, T_ny), d(T_nx, E_nx), d(T_nx, F_ny), \\ d(T_ny, E_nx), d(T_ny, F_ny)) + d(F_ny, Fy). \end{aligned}$$

Using uniform convergence of $\{E_n\}$, $\{F_n\}$, $\{T_n\}$, upper semicontinuity of h and inequalities of the form

$$d(T_n x, T_n y) \leqslant d(T_n x, Tx) + d(Tx, Ty) + d(Ty, T_n y),$$

we find that E, F and T satisfy condition (C_2). Note that T is continuous. Hence by Theorem 1, E, F and T have a unique common fixed point x_0. Convergence of $\{x_n\}$ to x_0 follows from Theorem 4. This proves the result.

Theorem 6 — Let $\{E_n\}$, $\{F_n\}$ and $\{T_n\}$ be sequences of self-mappings of a complete metric space (X, d) such that for each n, T_n is continuous, and E_n, F_n and T_n satisfy conditions (C_1) and (C_2) for each n. Let x_n be the unique common fixed point E_n, F_n and T_n for each n. Let E, F and T be self-mappings of X such that $E_n \rightarrow E$, $F_n \rightarrow F$ and $T_n \rightarrow T$. If x_0 is any cluster point of the sequence $\{x_n\}$ then

$$Ex_0 = Fx_0 = Tx_0 = x_0.$$

PROOF: As x_0 is a cluster point of the sequence $\{x_n\}$ we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow x_0$. Since T_{n_i} is continuous for each $i \geqslant 1$, $T_{n_i} x_{n_i} \rightarrow T_{n_i} x_0$. So $x_{n_i} \rightarrow T_{n_i} x_0$. This implies that $T_{n_i} x_0 = x_0$ for each $i \geqslant 1$. Since $T_{n_i} \rightarrow T$, therefore $d(T_{n_i} x_0, Tx_0) \rightarrow 0$. Then $d(x_0, Tx_0) = d(T_{n_i} x_0, Tx_0)$ says that $Tx_0 = x_0$.

Now for each $i \geqslant 1$, we have

$$d(x, Ex_0) \leqslant d(x_0, x_{n_i}) + d(F_{n_i} x_{n_i}, E_{n_i} x_0), + d(E_{n_i} x_0, Ex_0).$$

By (C_2) we have

$$\begin{aligned} d(E_{n_i} x_0, F_{n_i} x_{n_i}) &\leqslant h(d(T_{n_i} x_0, T_{n_i} x_{n_i}), d(T_{n_i} x_0, E_{n_i} x_0), d(T_{n_i} x_0, F_{n_i} x_{n_i}), \\ &\qquad d(T_{n_i} x_{n_i}, E_{n_i} x_0), d(T_{n_i} x_{n_i}, F_{n_i} x_{n_i})) \\ &= h(d(x_0, x_{n_i}), d(x_0, E_{n_i} x_0), d(x_0, x_{n_i}), \\ &\qquad d(x_{n_i}, E_{n_i} x_0), d(x_{n_i}, x_{n_i})). \end{aligned}$$

Furthermore, for each $i \geqslant 1$,

$$d(x_0, E_{n_i} x_0) \leqslant d(x_0, Ex_0) + d(Ex_0, E_{n_i} x_0)$$

and

$$d(x_{n_i}, E_{n_i} x_0) \leqslant d(x_{n_i}, x_0) + d(x_0, Ex_0) + d(Ex_0, E_{n_i} x_0).$$

Therefore, as $i \rightarrow \infty$ we get

$$d(x_0, Ex_0) \leqslant h(0, d(x_0, Ex_0), 0, d(x_0, Ex_0), 0)$$

which gives $x_0 = Ex_0$.

Similarly, we can prove that $Fx_0 = x_0$. Thus x_0 is a common fixed point of E, F and T, as required.

Remark : Taking $E_n = F_n$ for each n, and $T_n = I_x$ in Theorem 6, we get Theorem 4 of Husain and Sehgal (1975) which generalizes Theorem 3 of Iséki (1974).

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