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# ON $M$-VOID INTEGERS

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Let $M$ be a set of positive integers with least element $\geqslant 2$. We say a positive integer is $M$-void if in its canonical factorization as a product of prime powers, each exponent lies outside $M$. In the present paper, an asymptotic formula is derived for the number of representations of a positive integer as the sum of an $M_1$-void integer and an $M_2$-void integer.

## 1. INTRODUCTION

Let $M$ be a set of positive integers with minimal element $r \geqslant 2$. A positive integer is called $M$-void if in its canonical factorisation into prime powers, each exponent lies outside $M$. The concept of an $M$-void integer is recently introduced by Rieger (1973). Let $Q_M$ denote the set of all $M$-void integers and for each positive integer $n$, we write $T_{M_1,M_2}(n)$ to denote the number of pairs $(x, y) \in Q_{M_1} \times Q_{M_2}$ with $x + y = n$ where $M_1$, $M_2$ are sets of positive integers with $\min M_i \geqslant 2$, $i = 1, 2$. In this paper, we establish an asymptotic formula for $T_{M_1,M_2}(n)$ and deduce as special cases results due to Evelyn and Linfoot (1931), Estermann (1931) on $r$-free integers and a refinement of a recent result due to Brinitzer (1975) on $(k, r)$-integers (see section 4).

## 2. PRELIMINARIES

Let $\gamma_M(n)$ denote the characteristic function of $Q_M$ and $\lambda(n) = \lambda_M(n)$ the unique arithmetical function defined by $\gamma_M(n) = \sum_{d \mid n} \lambda_M(d)$, that is, $\lambda_M(n) = \sum_{d \mid n} \mu(d)\, \gamma_M(n/d)$, $\mu$ being the Möbius function. Clearly, $\lambda$ is multiplicative; for prime $p$ and positive integral $m$, $|\lambda(p^m)| = 1$ if $m \geqslant 2$ and exactly one of $m$ and $m - 1$ lies in $M$ and $\lambda(p^m) = 0$ otherwise. Writing $S = \{n \mid n \text{ is an integer} \geqslant r,\ n \notin M\}$, we put $k = k_M = \infty$ or $\min S$ according as $S$ is empty or not. For $s > \max\left(\frac{1}{2r}, \frac{1}{k}\right)$, we write

$$f(s) = f_M(s) = \prod_p \{1 + p^{-2rs} + (1 + p^{-rs}) \sum_{m=k}^{\infty} |\lambda(p^m)|\, p^{-ms}\}$$

the product ranging over all primes $p$. $\zeta(s)$ denotes the Riemann zeta function and $\tau(n)$, the number of positive divisors of $n$.

*Lemma* 2.1 — For $x \geqslant 1$ and $0 \leqslant \epsilon < \frac{1}{4}$, we have

(i) $\sum_{n<x} | \lambda(n) | = O(x^{1/r} f(r^{-1}))$,

(ii) $\sum_{n<x} | \lambda(n) | n^{-1+\epsilon} = O(x^{(1-r+r\epsilon)/r} f(r^{-1}))$,

the $O$-constants being absolute.

PROOF : For $s > 1$, we have by Euler's infinite product factorization Theorem (cf. Hardy and Wright 1968, Theorem 286)

$$\sum_{n=1}^{\infty} | \lambda(n) | n^{-s} = \prod_p \{1 + p^{-rs} + \sum_{m=k}^{\infty} | \lambda(p^m) | p^{-ms}\}$$

$$= \zeta(rs) \prod_p \{(1 - p^{-rs}) (1 + p^{-rs} + \sum_{m=k}^{\infty} | \lambda(p^m) | p^{-ms})\}$$

$$= \zeta(rs) \prod_p \{1 - p^{-2rs} + (1 - p^{-rs}) \sum_{m=k}^{\infty} | \lambda(p^m) | p^{-ms}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{rs}} \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \text{ say,}$$

where the series $\sum_{n=1}^{\infty} a(n)\, n^{-s}$ converges absolutely for $s > \max\left(\frac{1}{2r}, \frac{1}{k}\right)$ (cf. Estermann 1952, Theorem 41). Hence

$$\sum_{n<x} | \lambda(n) | \leqslant \sum_{n \leqslant x} \sum_{d^r \delta = n} | a(\delta) | \leqslant x^{1/r} \sum_{\delta < x} \frac{| a(\delta) |}{\delta^{1/r}}$$

$$\leqslant x^{1/r} \sum_{\delta=1}^{\infty} \frac{| a(\delta) |}{\delta^{1/r}} \leqslant x^{1/r} f(r^{-1}).$$

Hence (i) follows. By partial summation and (i)

$$\sum_{n>x} | \lambda(n) | n^{-1+\epsilon} = O(f(r^{-1})\, x^{(1-r+\epsilon r)/r})$$

$$+ O(\int_x^{\infty} f(r^{-1})\, t^{1/r}(1 - \epsilon)\, t^{-2+\epsilon}\, dt)$$

$$= O(f(r^{-1})\, x^{(1-r+\epsilon r)/r})$$

$$+ O\left(\frac{1 - \epsilon}{1 - \epsilon - r^{-1}} f(r^{-1})\, x^{(1-r+\epsilon r)/r}\right)$$

$$= O(x^{(1-r+\epsilon r)/r} f(r^{-1}))$$

since $0 \leqslant (1-\epsilon)/(1-\epsilon-r^{-1}) \leqslant 4r/(3r-4) \leqslant 4$. This completes the proof of the lemma.

*Lemma* 2.2 — For positive integral $n$

$$\sum_{m=1}^{\infty} \frac{|\lambda(m)|\,(m,n)}{m} \leqslant \frac{\zeta(2)\,\zeta(3)}{\xi(6)}\,\tau(n).$$

PROOF : By Euler's infinite product factorization theorem (Hardy and Wright 1968, Th. 286)

$$\sum_{m=1}^{\infty} \frac{|\lambda(m)|\,(m,n)}{m}$$

$$= \prod_{p \mid n} \left\{1 + \sum_{m=r}^{\infty} \frac{|\lambda(p^m)|}{p^m}\right\} \prod_{p^\alpha \parallel n} \left\{1 + \sum_{m=r}^{\infty} \frac{|\lambda(p^m)|\,p^{\min(\alpha,m)}}{p^m}\right\}$$

$$\leqslant \prod_{p} \left\{1 + \sum_{m=2}^{\infty} p^{-m}\right\} \prod_{\substack{p^\alpha \parallel n \\ \alpha \geqslant r}} \left\{1 + \sum_{m=r}^{\alpha} 1 + \sum_{m=\alpha+1}^{\infty} p^{\alpha-m}\right\}$$

$$\times \prod_{\substack{p^\alpha \parallel n \\ \alpha < r}} \left\{1 + p^\alpha \sum_{m=r}^{\infty} p^{-m}\right\}$$

$$= \prod_{p} \left\{1 + \frac{1}{p(p-1)}\right\} \prod_{\substack{p^\alpha \parallel n \\ \alpha \geqslant r}} \left(1 + \alpha - r + 1 + \frac{1}{p-1}\right)$$

$$\times \prod_{\substack{p^\alpha \parallel n \\ \alpha < r}} \left(1 + \frac{p^{\alpha-r+1}}{p-1}\right)$$

$$\leqslant \prod_{p} \left\{\frac{(1-p^{-6})}{(1-p^{-2})\,(1-p^{-3})}\right\} \prod_{\substack{p^\alpha \parallel n \\ \alpha \geqslant r}} (\alpha+1) \prod_{\substack{p^\alpha \parallel n \\ \alpha < r}} (2)$$

$$\leqslant \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} \prod_{p^\alpha \parallel n} (\alpha+1).$$

Hence the lemma follows.

## 3. Main Results

Let $M_1$ and $M_2$ be sets of positive integers with min $M_i = r_i \geqslant 2$, $i = 1, 2$. We write $\gamma_i(n)$, $\lambda_i(n)$ and $f_i(s)$ to mean respectively $\gamma_{M_i}(n)$, $\lambda_{M_i}(n)$ and $f_{M_i}(s)$;

$$r = \min(r_1, r_2), \quad \alpha = \alpha_{M_1,M_2} = \max(f_1(r_1^{-1}), f_2(r_2^{-1}))$$

and $$\beta = \beta_{M_1,M_2} = f_1(r_1^{-1})\, f_2(r_2^{-1}).$$

Then we prove the following.

*Theorem* — For each $\epsilon > 0$, we have, as $n \to \infty$

$$T_{M_1,M_2}(n) = nH_{M_1,M_2}(n) + O_\epsilon(\nu_{M_1,M_2,\epsilon}\, n^A),$$

$$\left[\text{where } A = \frac{r(r_1 + r_2)}{r_1 r_2(r-1) + r(r_1 + r_2)} + \epsilon\right] \quad \ldots(3.1)$$

where

$$H_{M_1,M_2}(n) = \sum_{\substack{u, v=1 \\ (u,v) \mid n}}^{\infty} \frac{\lambda_1(u)\, \lambda_2(v)\, (u, v)}{uv} \quad \ldots(3.2)$$

and

$$\nu_{M_1,M_2,\epsilon} = \{\alpha^{r(r_1+r_2)} \beta^{r_1 r_2(r-1)}\}^B,$$

$$\left[\text{where } B = \frac{1}{r_1 r_2(r-1) + r(r_1 + r_2)} + \epsilon\right]. \quad \ldots(3.3)$$

Proof: Without loss of generality we can assume that $0 < \epsilon < \frac{1}{4}$. We have

$$T_{M_1,M_2}(n) = \sum_{x+y=n} \gamma_1(x)\, \gamma_2(y) = \sum_{x<n} \gamma_1(x)\, \gamma_2(n-x)$$

$$= \sum_{x<n} \Big(\sum_{\substack{u \mid x \\ u \leqslant t}} \lambda_1(u)\Big)\Big(\sum_{\substack{v \mid n-x \\ v \leqslant t}} \lambda_2(v)\Big) + \sum_{x<n} \sum_{\substack{u \mid x \\ u>t}} \lambda_1(u)\, \gamma_2(n-x)$$

$$+ \sum_{x<n} \gamma_1(x) \sum_{\substack{v \mid n-x \\ v>t}} \lambda_2(v) - \sum_{x<n} \Big(\sum_{\substack{u \mid x \\ u>t}} \lambda_1(u)\Big)\Big(\sum_{\substack{v \mid n-x \\ v>t}} \lambda_2(v)\Big)$$

$$= T_1 + T_2 + T_3 - T_4, \text{ say} \quad \ldots(3.4)$$

where $1 \leqslant t = t(n) \leqslant n$ and the function $t(n)$ will be chosen suitably later. If $b_1$, $b_2$ are integers and $a_1$, $a_2$ are positive integers, it is known that the system of congruences $x \equiv b_i \pmod{a_i}$, $i = 1, 2$ is solvable iff the g.c.d. $(a_1, a_2) \mid b_1 - b_2$ and that in case of solvability, the system admits a unique solution modulo the l.c.m. $\{a_1, a_2\}$. Hence by (i) of Lemma 2.1,

$$T_1 = \sum_{\substack{u<t,\, v<t \\ (u,v)\mid n}} \lambda_1(u)\,\lambda_2(v) \Bigg( \sum_{\substack{x<n,\, x\equiv O \,(\mathrm{mod}\, u) \\ x\equiv n \,(\mathrm{mod}\, v)}} 1 \Bigg)$$

$$= \sum_{\substack{u<t,\, v<t \\ (u,v)\mid n}} \lambda_1(u)\,\lambda_2(v) \left( \frac{n}{\{u, v\}} + O(1) \right)$$

$$= n \sum_{\substack{u<t,\, v<t \\ (u,v)\mid n}} \frac{\lambda_1(u)\,\lambda_2(v)\,(u, v)}{uv} + O\Bigg(\Bigg( \sum_{u<t} |\lambda_1(u)| \Bigg)\Bigg( \sum_{v<t} |\lambda_2(v)| \Bigg)\Bigg)$$

$$= nT_{11} + O(\beta t^{(r_1+r_2)/r_1 r_2}). \qquad \ldots(3.5)$$

We write

$$T_{11} = \sum_{\substack{u,v=1 \\ (u,v)\mid n}} \frac{\lambda_1(u)\,\lambda_2(v)\,(u, v)}{uv} - \sum_{\substack{u\leqslant t,\, v>t \\ (u,v)\mid n}} \cdots - \sum_{\substack{u>t,\, v\leqslant t \\ (u,v)\mid n}} \cdots + \sum_{\substack{u>t,\, v>t \\ (u,v)\mid n}} \cdots$$

$$= T_{11}^{(1)} - T_{11}^{(2)} - T_{11}^{(3)} + T_{11}^{(4)}, \text{ say.} \qquad \ldots(3.6)$$

By Lemma 2.2 and (ii) of Lemma 2.1, we have

$$T_{11}^{(2)} = O\Bigg( \sum_{v>t} \frac{|\lambda_2(v)|}{v} \sum_{u=1}^{\infty} \frac{|\lambda_1(u)|\,(u, v)}{u} \Bigg)$$

$$= O\Bigg( \sum_{v>t} \frac{|\lambda_2(v)|\,\tau(v)}{v} \Bigg)$$

$$= O_\epsilon\Bigg( \sum_{v>t} \frac{|\lambda_2(v)|}{v^{1-\epsilon}} \Bigg)$$

$$= O_\epsilon(\alpha t^{(1-r+\epsilon r)/r})$$

where we used the fact that $\tau(n) = O_\delta(n^\delta)$ for each $\delta > 0$ (cf. Hardy and Wright 1968, Theorem 315). Similarly each one of $T_{11}^{(3)}$ and $T_{11}^{(4)}$ is $O_\epsilon(\alpha t^{(1-r+\epsilon r)/r})$ so that by (3.5) and (3.6)

$$T_1 = nH_{M_1,M_2}(n) + O_\epsilon(\alpha n t^{(1-r+\epsilon r)/r}) + O(\beta t^{(r_1+r_2)/r_1 r_2}) \qquad \ldots(3.7)$$

where $H_{M_1,M_2}(n)$ is as given in (3.2). Also by (ii) of Lemma 2.1

$$T_2 = O\Big(\sum_{x<n} \sum_{\substack{ud=x \\ u>t}} |\lambda_1(u)|\Big) = O\Big(n \sum_{u>t} \frac{|\lambda_1(u)|}{u}\Big)$$

$$= O(\alpha n t^{(1-r)/r}). \qquad \ldots(3.8)$$

Similarly

$$T_3 = O(\alpha n t^{(1-r)/r}). \qquad \ldots(3.9)$$

By Cauchy-Schwarz inequality

$$T_4 = O\Big(\sum_{x<n}\Big(\sum_{\substack{u \mid x \\ u>t}} |\lambda_1(u)|\Big)\Big(\sum_{\substack{v \mid n-x \\ v>t}} |\lambda_2(v)|\Big)\Big)$$

$$= O\Big(\Big[\sum_{x<n}\Big(\sum_{\substack{u \mid x \\ u>t}} |\lambda_1(u)|\Big)^2\Big]^{1/2}\Big[\sum_{x<n}\Big(\sum_{\substack{v \mid n-x \\ v>t}} |\lambda_2(v)|\Big)^2\Big]^{1/2}\Big)$$

$$= O\Big(\max_{i=1,2}\Big[\sum_{x<n}\Big(\sum_{\substack{u \mid x \\ u>t}} |\lambda_i(u)|\Big)^2\Big]\Big)$$

$$= O\Big(\max_{i=1,2}\Big[\sum_{u>t,\, v>t} |\lambda_i(u)|\,|\lambda_i(v)| \sum_{\substack{x<n \\ x\equiv O(\mathrm{mod}\,\{u,v\})}} 1\Big]\Big)$$

$$= O\Big(\max_{i=1,2}\Big[n \sum_{u>t,\, v>t} \frac{|\lambda_i(u)|\,|\lambda_i(v)|\,(u,v)}{uv}\Big]\Big)$$

$$= O_\epsilon(\alpha n t^{(1-r+r\epsilon)/r}) \qquad \ldots(3.10)$$

as in the estimation of $T_{11}^{(2)}$ above. Thus by (3.4) and (3.7) through (3.10) we have

$$T_{M_1,M_2}(n) = nH_{M_1,M_2}(n) + O_\epsilon(\alpha n t^{(1-r+r\epsilon)/r}) + O(\beta t^{(r_1+r_2)/r_1 r_2}).$$

Now choosing $t = t(n) = (\alpha\beta^{-1}n)^{r_1 r_2 r/[r_1 r_2(r-1)+r(r_1+r_2)]}$ we see that each of the above two $O$-terms reduces to

$$O_\epsilon(\nu_{M_1,M_2,\epsilon}\, n^{[r(r_1+r_2)/(r_1 r_2(r-1)+r(r_1+r_2))]+\epsilon})$$

where $\nu_{M_1,M_2,\epsilon}$ is as given in (3.3). This completes the proof of the theorem.

## 4. Applications

In this section, we illustrate our theorem of section 3 by specializing the sets $M_1$ and $M_2$.

Let $r$, $r_1$, $r_2$, $k$, $k_1$ and $k_2$ be integers $\geqslant 2$ and $s$ a positive integer. We write

$$M^{(1)}(r) = \{n \mid n \text{ is integral}, n \geqslant r\};$$

$$M^{(2)}(k, r) = \left\{\begin{matrix} n \mid n \geqslant r \text{ and } n \text{ is congruent to at least} \\ \text{one of } r, r+1, \ldots, k-1 \pmod{k} \end{matrix}\right\} \text{ where } r < k;$$

$$M^{(3)}(s, r) = \{r, 2r, \ldots, sr\};$$

$$M^{(4)}(r) = \{r, 2r, 3r, \ldots\};$$

$$M^{(5)}(r) = \{r\}.$$

The sets $Q_{M^{(1)}(r)}, \ldots, Q_{M^{(5)}(r)}$ will be denoted respectively by

$$Q_r, Q_{k,r}, Q^*_{s,r}, Q^*_r, Q^{s*}_r$$

and elements of these sets will be respectively called $r$-free integers, $(k, r)$-integers (Subba Rao and Harris 1966), unitarily $(s, r)$-integers, unitarily $r$-free integers (Cohen 1961, 1964), semi-$r$-free integers (Suryanarayana 1971). It may be noted that $Q^{s*}_r = Q^*_{1,r}$ and thus the notion of semi-$r$-free integers is essentially contained in Cohen's works (1961, 1964). Further for each positive integer $n$, we write $T_r(n)$, $T_{r_1,r_2}(n)$, $T_{(k_1,r_1),(k_2,r_2)}(n)$ and $T^*_{r,r}(n)$ respectively to mean the number of pairs $(x, y)$ of integers with $x + y = n$ and $(x, y) \in Q_r \times Q_r$, $(x, y) \in Q_{r_1} \times Q_{r_2}$,

$$(x, y) \in Q_{(k_1,r_1)} \times Q_{(k_2,r_2)},\ (x, y) \in Q_r \times Q^{s*}_r\,.$$

In the theorem of section 3, taking (a) $M_1 = M_2 = M^{(1)}(r)$, (b) $M_1 = M^{(1)}(r_1)$, $M_2 = M^{(1)}(r_2)$ where $r_1 \leqslant r_2$, (c) $M_1 = M^{(2)}(k_1, r_1)$, $M_2 = M^{(2)}(k_2, r_2)$ where $r_1 < k_1$ and $r_2 < k_2$ and (d) $M_1 = M^{(1)}(r)$, $M_2 = M^{(5)}(r)$ in turn, we obtain the following:

For each $\epsilon > 0$ and as $n \to \infty$

$$T_r(n) = \prod_p \left(1 - \frac{2}{p^r}\right) \prod_{p^r/n} \left(1 + \frac{1}{p^r - 2}\right) n + O\left(n^{[2/(r+1)]+\epsilon} \left(\frac{15}{\pi^2}\right)^{2r^3\epsilon}\right) \qquad \ldots(4.1)$$

$$T_{r_1,r_2}(n) = \prod_p \left(1 - \frac{1}{p^r} - \frac{1}{p^{r_2}}\right) \prod_{p^{r_1}/n} \left(1 + \frac{1}{p^{r_2} - p^{r_2 - r_1} - 1}\right) n + O(n^{[(r_1+r_2)/r_1(r_2+1)]+\epsilon}) \qquad \ldots(4.2)$$

$$T_{(k_1,r_1),(k_2,r_2)}(n) = \frac{\zeta(k_2)}{\zeta(r_2)} \prod_p \left(1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \cdot \frac{p^{-k_1} - p^{-r_1}}{1-p^{-k_1}}\right) nB(n)$$
$$+ O\left(\nu_{k_1,r_1;k_2,r_2;\epsilon} n^{A}\right),$$
$$\left[\text{where } A = \frac{r(r_1+r_2)}{r_1 r_2 (r-1) + r(r_1+r_2)} + \epsilon\right] \qquad \ldots(4.3)$$

where

$$B(n) = \prod_{p \mid n} \left\{1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \cdot \frac{p^{-k_1} - p^{-r_1}}{1-p^{-k_1}}\right\}^{-1}$$
$$\times \left[1 + \frac{1-p^{-k_2}}{1-p^{-r_2}} \left\{ \sum_{\mu=1}^{\infty} \frac{1}{p^{\mu k_1}} \left( \sum_{\substack{m=0 \\ (p^{mk_2}, p^{\mu k_1})/n}}^{\infty} \frac{(p^{mk_2}, p^{\mu k_1})}{p^{mk_2}} \right.\right.\right.$$
$$\left. - \sum_{\substack{m=0 \\ (p^{r_2+mk_2}, p^{\mu k_1})/n}}^{\infty} \frac{(p^{r_2+mk_2}, p^{\mu k_1})}{p^{r_2+mk_2}} \right)$$
$$- \sum_{\mu=0}^{\infty} \frac{1}{p^{r_1+\mu k_1}} \left( \sum_{\substack{m=0 \\ (p^{mk_2}, p^{r_1+\mu k_1})/n}}^{\infty} \frac{(p^{mk_2}, p^{r_1+\mu k_1})}{p^{mk_2}} \right.$$
$$\left.\left.\left. - \sum_{\substack{m=0 \\ (p^{r_2+mk_2}, p^{r_1+\mu k_1})/n}}^{\infty} \frac{(p^{r_2+mk_2}, p^{r_1+\mu k_1})}{p^{r_2+mk_2}} \right)\right\}\right]$$

$$r = \min(r_1, r_2)$$

$$\nu_{k_1,r_1;k_2,r_2;\epsilon} = O\left\{\left(\alpha^{r(r_1+r_2)} \beta^{r_1 r_2 (r-1)}\right)^{A'}\right\},$$
$$\left[\text{where } A' = \frac{1}{r_1 r_2 (r-1) + r(r_1+r_2)} + \epsilon\right]$$

$$\alpha = \alpha_{k_1,r_1;k_2,r_2} \equiv \prod_p \left(1 + \frac{3}{p^2}\right) \max\left(\zeta\left(\frac{k_1}{r_1}\right), \zeta\left(\frac{k_2}{r_2}\right)\right)$$

$$\beta = \beta_{k_1,r_1;k_2,r_2} \equiv \prod_p \left(1 + \frac{3}{p^2}\right)^2 \zeta\left(\frac{k_1}{r_1}\right) \zeta\left(\frac{k_2}{r_2}\right)$$

$$T^*_{r,r}(n) = \prod_{p \mid n} \left(1 - \frac{1}{p^r}\right) \prod_{p \nmid n} \left(1 - \frac{2}{p^r} + \frac{1}{p^{r+1}}\right) n + O\left(n^{[2/(r+1)]+\epsilon}\right)$$
$$\ldots(4.4)$$

where the order constants in (4.1) and (4.3) depend only on $\epsilon$ while the order constants depend only on $\epsilon$ and $r$ in (4.2) and (4.4).

*Remark* 1 : It may be noted that in each of the above four examples of our theorem, the corresponding $H_{M_1,M_2}(n)$, as given by (3.2), is given an alternate form by appealing to Euler's infinite product factorization theorem (Hardy and Wright 1968, Th. 286). The verifications are straightforward.

*Remark* 2 : Evelyn and Linfoot (1931) established (4.1) while Estermann (1931) gave a purely elementary proof of it. Recently, the authors (Rao and Murty 1979) showed that the order of the error term on the right of (4.1) is $O_r((n \log\log 3n)^{2/(r+1)})$ which refines the earlier results due to Cohen (1965), in case $r = 2$, Pomerance and Suryanarayana (1979) and the authors (Rao and Murty 1979). It may be noted that the $O$-constant on the right of (4.1) is independent of $r$ which was not made out explicitly by the earlier authors.

*Remark* 3 : Clearly (4.1) is a special case of (4.2). Page (1932) establised a stronger form of (4.2), namely, that the $O$-term on the right of (4.2) is

$$O_{r_1,r_2,\epsilon}(n^{[(r_1+r_2-2)/(r_1r_2-1)]+\epsilon}).$$

Recently, Rao and Murty (1979a) and Pomerance and Suryanarayana (1979) independently improved the $O$-estimate of the error term in Page's formula to

$$O_{r_1,r_2}(n^{(r_1+r_2)/r_2(r_1+1)} (\log 2n)^Z)$$

$$\left[Z = (r_1^{(1/(r_1+1))} - 1)\left(\frac{r_2 - 1}{r_2}\right)\right]$$

if $r_2 \leqslant r_1^2$ and to $O_{r_1,r_2}(n^{1/r_1})$ if $r_2 > r_1^2$ ·

*Remark* 4 : A weak form of (4.3) was recently established by Brinitzer (cf. Brinitzer 1975, Satz 3). However, the following changes should be incorporated in Brinitzer (1975) (according to a private communication to the authors by Dr E. Brinitzer). The conclusion of Satz 1 of Brinitzer should read

$$Q_{k,r}(x; a, b) = \frac{\zeta(k)}{\zeta(r)}\frac{x}{b}\prod_{p \mid b}\left(\frac{1-p^{-k}}{1-p^{-r}}\right)$$

$$\times \prod_{p \mid (a,b)}\left\{\sum_{\substack{m=0\\(p^{mk},b)/a}}^{\infty}\frac{(p^{mk},b)}{p^{mk}} - \sum_{\substack{m=0\\(p^{r+mk},b)/a}}^{\infty}\frac{(p^{r+mk},b)}{p^{r+mk}}\right\}$$

$$+ O(\zeta(k/r)\, x^{1/r}).$$

And consequently, the main term on the right of (1.5) of Brinitzer (1975) should be replaced by the main term appearing on the right hand of (4.3) above. We also note that a special case of Satz 3 of Brinitzer (1975) was earlier attempted by Subba Rao and Feng (1971, Theorem 2). However, in their argument the transition from the last two lines of page 746 to the first three lines of page 747 could not be justified since in the latter, the sum $\sum_{\substack{m \leqslant n^{(1/k_1)-\varepsilon} \\ (m^{k_1}, n) \notin Q_{k_2}, \gamma_2}} \lambda(m)\, Q^{*(n;n,mk_1)}_{(k_2 r_2, k_2 \gamma_2)}$ was not accounted for. This invalidates their result.

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