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FIXED POINT THEOREMS FOR OPERATORS ON BANACH SPACES

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The purpose of this note is to present a generalization of a theorem of Bernfeld *et al.* (1977) and also to give a valid proof of another theorem of the same authors.

§1. Let E be a Banach space and E_0 be the Banach space of all continuous functions from a finite closed interval $[a, b]$ into E where

$$\| f \|_{E_0} = \sup_{a \leqslant t \leqslant b} \| f(t) \|_E \;\forall\; f \in E_0.$$

Let T be an operator from E_0 into E. We shall say that $f \in E_0$ is a fixed point of T if $Tf = f(c)$ for some fixed $c \in [a, b]$. T is defined to be a contraction if

$$\| Tf - Tg \|_E \leqslant \alpha \| f - g \|_{E_0} \;\forall\; f, g \in E_0, \quad 0 \leqslant \alpha < 1.$$

Bernfeld *et al.* (1977) proved the following:

Theorem 1 — Suppose $T : E_0 \to E$ is a contraction. Then the following hold:

(i) Given $f_0 \in E_0$, every sequence of iterates $\{f_n\}$ satisfying $Tf_n = f_{n+1}(c)$, for a given $c \in [a, b]$ and $\| f_{n+1} - f_n \|_{E_0} = \| f_{n+1}(c) - f_n(c) \|_E$ converges to a fixed point f^* of T.

(ii) Given $f_0, g_0 \in E_0$, let $\{f_n\}$ and $\{g_n\}$ be the sequences of iterates corresponding to f_0 and g_0 constructed as in (i). Then

$$\| f_n - g_n \|_{E_0} \leqslant \frac{1}{1 - \alpha} \{ \| f_1 - f_0 \|_{E_0} + \| g_1 - g_0 \|_{E_0} \} + \| f_0 - g_0 \|_{E_0}$$

If, in particular, $f_0 = g_0$ and $\{f_n\} \not\equiv \{g_n\}$, then

$$\| f_n - g_n \|_{E_0} \leqslant \frac{2}{1 - \alpha} \| f_1 - f_0 \|_{E_0}.$$

(iii) Let $\Omega_0 = \{ f \in E_0 / \| f \|_{E_0} = \| f(c) \|_E \}$ and let $\{f_n\}, \{g_n\}$ be as in (ii). If $f_n - g_n \in \Omega_0$ for all n, then $\lim f_n = \lim g_n$. Finally, if we define

$$\Omega_{f^*} = \{ f \in E_0 / \| f - f^* \|_{E_0} = \| f(c) - f^*(c) \|_E \}$$

where f^* is a fixed point of T, then f^* is the only fixed point of T in Ω_{f^*}.

We now prove the following theorem that generalizes Theorem 1.

Theorem 2 — Suppose $T : E_0 \to E$ is such that $\forall\ f, g \in E_0$

$$\| Tf - Tg \|_E \leqslant \alpha \| f - g \|_{E_0} + \beta \{ \| f(c) - Tf \|_E + \| g(c) - Tg \|_E \} + \gamma \{ \| f(c) - Tg \|_E + \| g(c) - Tf \|_E \}$$

where $\alpha, \beta, \gamma \geqslant 0$, $\alpha + 2\beta + 2\gamma < 1$. Then, the conclusions (i), (ii) and (iii) of Theorem 1 hold, with the only change that α in (ii) is to be replaced by

$$\delta = \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}.$$

PROOF: Let $f_0 \in E_0$ be given. By hypothesis, $Tf_0 \in E$. Choose $f_1 \in E_0$ such that $Tf_0 = f_1(c)$ and $\| f_1(c) - f_0(c) \|_E = \| f_1 - f_0 \|_{E_0}$. Defining f_n inductively so that $Tf_n = f_{n+1}(c)$ and

$$\| f_{n+1}(c) - f_n(c) \|_E = \| f_{n+1} - f_n \|_{E_0} \quad \text{for} \quad n = 1, 2, \ldots$$

$$\begin{aligned} \| f_n - f_{n+1} \|_{E_0} &= \| f_n(c) - f_{n+1}(c) \|_E \\ &= \| Tf_{n-1} - Tf_n \|_E \\ &\leqslant \alpha \| f_{n-1} - f_n \|_{E_0} + \beta \{ \| f_{n-1}(c) - Tf_{n-1} \|_E + \| f_n(c) - Tf_n \|_E \} \\ &\quad + \gamma \{ \| f_{n-1}(c) - Tf_n \|_E + \| f_n(c) - Tf_{n-1} \|_E \} \\ &= \alpha \| f_{n-1} - f_n \|_{E_0} + \beta \{ \| f_{n-1}(c) - f_n(c) \|_E \\ &\quad + \| f_n(c) - f_{n+1}(c) \|_E \} + \gamma \| f_{n-1}(c) - f_{n+1}(c) \|_E \\ &= \alpha \| f_{n-1} - f_n \|_{E_0} + \beta \{ \| f_{n-1} - f_n \|_{E_0} + \| f_n - f_{n+1} \|_{E_0} \} \\ &\quad + \gamma \{ \| f_{n-1} - f_n \|_{E_0} + \| f_n - f_{n+1} \|_{E_0} \} \end{aligned}$$

i.e.,

$$\begin{aligned} \| f_n - f_{n+1} \|_{E_0} &\leqslant \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \| f_n - f_{n-1} \|_{E_0} \\ &= \delta \| f_n - f_{n-1} \|_{E_0}, \quad \delta < 1. \end{aligned}$$

This gives us

$$\| f_n - f_{n+1} \|_{E_0} \leqslant \delta^n \| f_0 - f_1 \|_{E_0}.$$

If $m \geqslant n$, by repeated application of the triangle inequality, we get

$$\begin{aligned} \| f_m - f_n \|_{E_0} &\leqslant \| f_m - f_{m-1} \|_{E_0} + \| f_{m-1} - f_{m-2} \|_{E_0} + \ldots + \| f_{n+1} - f_n \|_{E_0} \\ &\leqslant (\delta^{m-1} + \delta^{m-2} + \ldots + \delta^n) \| f_0 - f_1 \|_{E_0} \\ &\leqslant \frac{\delta^n}{1 - \delta} \| f_0 - f_1 \|_{E_0} \end{aligned}$$

This shows that $\{f_n\}$ is a Cauchy sequence. Hence $\exists\, f^* \in E_0$ such that $\lim_{n\to\infty} f_n = f^*$.

Now,

$$\begin{aligned}\| Tf^* - Tf_n \|_E \leqslant \alpha \| f^* - f_n \|_{E_0} &+ \beta \{ \| f^*(c) - Tf^* \|_E \\ &\qquad + \| f_n(c) - Tf_n \|_E \} \\ &+ \gamma \{ \| f^*(c) - Tf_n \|_E + \| f_n(c) - Tf^* \|_E \}\end{aligned}$$

i.e.

$$\begin{aligned}\| Tf^* - f_{n+1}(c) \|_E \leqslant \alpha \| f^* - f_n \|_{E_0} &+ \beta \{ \| f^*(c) - Tf^* \|_E \\ &\qquad + \| f_n(c) - f_{n+1}(c) \|_E \} \\ &+ \gamma \{ \| f^*(c) - f_{n+1}(c) \|_E + \| f_n(c) - Tf^* \|_E \}.\end{aligned}$$

Taking limits as $n \to \infty$, we have

$$\| Tf^* - f^*(c) \| \leqslant (\beta + \gamma) \| Tf^* - f^*(c) \|$$

Since $0 \leqslant \beta + \gamma < 1/2$, we have $Tf^* = f^*(c)$.

i.e., f^* is a fixed point of T. This proves (i).

To prove (ii), we find that

$$\begin{aligned}\| f_n - g_n \|_{E_0} &\leqslant \| f_n - f_{n-1} \|_{E_0} + \| f_{n-1} - g_{n-1} \|_{E_0} + \| g_{n-1} - g_n \|_{E_0} \\ &\leqslant \delta^{n-1} \| f_1 - f_0 \|_{E_0} + \| f_{n-1} - g_{n-1} \|_{E_0} + \delta^{n-1} \| g_1 - g_0 \|_{E_0} \\ &= \delta^{n-1} \{ \| f_1 - f_0 \|_{E_0} + \| g_1 - g_0 \|_{E_0} \} + \| f_{n-1} - g_{n-1} \|_{E_0}\end{aligned}$$

consequently by induction we have

$$\begin{aligned}\| f_n - g_n \|_{E_0} &\leqslant \delta^{n-1} \{ \| f_1 - f_0 \|_{E_0} + \| g_1 - g_0 \|_{E_0} \} \\ &\qquad + \delta^{n-2} \{ \| f_1 - f_0 \|_{E_0} + \| g_1 - g_0 \|_{E} \} \\ &\qquad + \| f_{n-2} - g_{n-2} \|_{E_0} \\ &\leqslant (\delta^{n-1} + \delta^{n-2} + \dots + 1) \{ \| f_1 - f_0 \|_{E_0} + \| g_1 - g_0 \|_{E_0} \} \\ &\qquad + \| f_0 - g_0 \|_{E_0} \\ &\leqslant (1 - \delta)^{-1} \{ \| f_1 - f_0 \|_{E_0} + \| g_1 - g_0 \|_{E} \} + \| g_0 - f_0 \|_{E_0}\end{aligned}$$

In particular, if $g_0 = f_0$, then $g_0(c) = f_0(c)$ and $Tg_0 = Tf_0$ which implies $g_1(c) = f_1(c)$. Hence

$$\begin{aligned}\| f_n - g_n \|_{E_0} &\leqslant 2(1 - \delta)^{-1} \| f_1(c) - f_0(c) \|_E \\ &= 2(1 - \delta)^{-1} \| f_1 - f_0 \|_{E_0}, \qquad \text{proving (ii).}\end{aligned}$$

To prove (iii), let $\{f_n\}$ and $\{g_n\}$ be as in (ii) and $f_n - g_n \in \Omega_0$ for $n = 0, 1, 2 \dots$. Then

$$\begin{aligned}\| f_n - g_n \|_{E_0} &= \| f_n(c) - g_n(c) \|_E \\ &= \| Tf_{n-1} - Tg_{n-1} \|_E \\ &\leqslant \alpha \| f_{n-1} - g_{n-1} \|_{E_0} + \beta \{ \| f_{n-1}(c) - Tf_{n-1} \|_E \\ &\quad + \| g_{n-1}(c) - Tg_{n-1} \|_E \} \\ &\quad + \gamma \{ \| f_{n-1}(c) - Tg_{n-1} \|_E + \| g_{n-1}(c) - Tf_{n-1} \|_E \} \\ &= \alpha \| f_{n-1} - g_{n-1} \|_{E_0} + \beta \{ \| f_{n-1}(c) - f_n(c) \|_E \\ &\quad + \| g_{n-1}(c) - g_n(c) \|_E \} \\ &\quad + \gamma \{ \| f_{n-1}(c) - g_n(c) \|_E + \| g_{n-1}(c) - f_n(c) \|_E \} \\ &\leqslant \alpha \| f_{n-1} - g_{n-1} \|_{E_0} + \beta \{ \| f_{n-1} - f_n \|_{E_0} + \| g_{n-1} - g_n \|_{E_0} \} \\ &\quad + \gamma \{ \| f_{n-1} - f_n \|_{E_0} + \| g_{n-1} - g_n \|_{E_0} + 2 \| f_n - g_n \|_{E_0} \}\end{aligned}$$

Since $f_n \to f^*$ and $g_n \to g^*$ where f^*, g^* are fixed points of T, we have on taking limits,

$$\| f^* - g^* \|_{E_0} \leqslant \alpha \| f^* - g^* \|_{E_0} + 2\gamma \| f^* - g^* \|_{E_0}$$

i.e., $\| f^* - g^* \| = 0$, since $1 - \alpha - 2\gamma > 0$

i.e., $f^* = g^*$.

Finally, if $g^*(\neq f^*)$ is a fixed point of T in Ωf^*, then

$$\begin{aligned}\| f^* - g^* \|_{E_0} &= \| f^*(c) - g^*(c) \|_E \\ &= \| Tf^* - Tg^* \|_E \\ &\leqslant \alpha \| f^* - g^* \|_{E_0} + \beta \{ \| f^*(c) - Tf^* \|_E \\ &\quad + \| g^*(c) - Tg^* \|_E \} \\ &\quad + \gamma \{ \| f^*(c) - Tg^* \|_E + \| g^*(c) - Tf^* \|_E \}\end{aligned}$$

i.e., $$\| f^* - g^* \|_{E_0} \leqslant (\alpha + 2\gamma) \| f^* - g^* \|_{E_0}$$

which implies $f^* = g^*$, since $0 \leqslant \alpha + 2\gamma < 1$.

This completes the proof.

§2. Before we take up the next theorem, we need the following:

Definition — Let $F(T) = \{ f^* \in E_0 : Tf^* = f^*(c) \}$ be non-empty. We shall say that T is quasi-nonexpansive if

$$\| Tf - f^*(c) \|_E \leqslant \| f - f^* \|_{E_0} \ \forall \ f \in E_0, \ \forall \ f^* \in F(T).$$

Bernfeld *et al.* (1977) stated the following theorem on the convergence of iterates of quasi-nonexpansive maps which extends a result of Petryshyn and Williamson (1973).

Theorem 3 — Suppose that $T : E_0 \to E$ is a quasi-nonexpansive operator. If $\{f_n\}$ is a sequence of iterates as in Theorem 1 for a given $f_0 \in E_0$ satisfying

$$\| f_n - f \|_{E_0} = \| f_n(c) - f(c) \|_E$$

for any $f \in F(T)$, then $\{f_n\}$ converges to a fixed point of T, if and only if

$$\lim_{n\to\infty} d_{E_0}(f_n, F(T)) = 0$$

It is well known that a quasi-nonexpansive operator need not be continuous and the proof of the above theorem as given in Bernfeld *et al.* (1977) makes use of the continuity of T, which is not a part of the hypotheses. We present below a valid proof of the theorem.

PROOF : Necessity part follows exactly as in Bernfeld *et al.* (1977). To prove the sufficiency of the condition, assume $\lim_{n\to\infty} d_{E_0}(f_n, F(T)) = 0$. It can easily be shown that $\{f_n\}$ is a Cauchy sequence. Hence $f_n \to f^*$ in E_0. We also have

$$\lim_{n\to\infty} d_{E_0}(f_n, F(T)) = d_{E_0}(f^*, F(T)) = 0.$$

This gives that f^* is a limit point of $F(T)$ and hence there exists a sequence of distinct points $\{p_n\}$ in $F(T)$ such that $p_n \to f^*$

$$\| Tf^* - p_n(c) \|_E \leqslant \| f^* - p_n \|_{E_0}.$$

Taking limits as $n \to \infty$, we have

$$\| Tf^* - f^*(c) \| \leqslant 0 \quad \text{i.e., } Tf^* = f^*(c).$$

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