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REMARKS ON TWO THEOREMS OF PETRYSHYN AND WILLIAMSON

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It is shown that the assumption of continuity of the map T in Theoıems 1.1 and 1.2 of Petryshyn and Williamson (1973) is redundant.

Petryshyn and Williamson (1973) proved the following theorem.

Theorem 1 — Let D be a closed subset of a Banach space X and let T map D continuously into X such that

(1.1) $F(T)$ (the set of fixed points of T) $\neq \phi$;

(1.2) for each $x \in D$ and every $p \in F(T)$

$$\| Tx - p \| \leqslant \| x - p \|;$$

(1.3) there exists an $x_0 \in D$ such that

$$x_n = T^n(x_0) \in D \quad \text{for each} \quad n \geqslant 1.$$

Then $\{x_n\}$ converges to a fixed point of T in D if and only if $\lim_{n \to \infty} d(x_n, F(T)) = 0$.

We now show that the above theorem is valid even if T is not assumed to be continuous.

Clearly the condition $\lim_{n \to \infty} d(x_n, F(T)) = 0$ is necessary. For the sufficiency, assume $\lim_{n \to \infty} d(x_n, F(T)) = 0$. It follows that $\{x_n\}$ is a Cauchy sequence, exactly as in Petryshyn and Williamson (1973). Hence $\{x_n\}$ converges to some $x^* \in D$, since D is closed. It is easy to see that $d(x^*, F(T)) = \lim_{n \to \infty} d(x_n, F(T) = 0$. This gives that x^* is a limit point of $F(T)$. Hence there exists a sequence $\{p_n\}$ of points in $F(T)$ such that $p_n \to x^*$. Now, $\| Tx^* - p_n \| \leqslant \| x^* - p_n \|$. Taking limits as $n \to \infty$

$$\| Tx^* - x^* \| \leqslant 0 \quad \text{i.e.,} \quad Tx^* = x^*.$$

Remark : As the proof of Theorem 1.2 in Petryshyn and Williamson (1973) depends on Theorem 1.1 therein, it follows immediately that Theorem 1.2 is also valid without the assumption of continuity of T.

Reference

Petryshyn, W. V., and Williamson, T. E. (1973). Strong and weak convergence of sequence of successive approximations for quasi-nonexpansive mappings. *J. Math. Anal. Applic.*, **43**, 459-97.