Indian J. pure appl. Math., 12(3) : 318–331, March 1981

LOWER BOUNDS ON SOME INTEGRAL INEQUALITIES IN n INDEPENDENT VARIABLES

V. M. SINGARE AND B. G. PACHPATTE

Department of Mathematics and Statistics, Marathwada University, Aurangabad 431004 (*Maharashtra*)

(*Received* 28 *May* 1980)

The aim of the present paper is to establish some integral inequalities in n independent variables which can be used as handy tools in obtaining the lower bounds on the solutions of a class of hyperbolic partial differential and integral equations.

1. INTRODUCTION

An enormous amount of attention has been given to the study of various problems in the theory of partial differential and integral equations by using some kinds of inequalities and variational principles. Excellent surveys of the work done prior to 1970 together with many references are contained in the monographs by Walter (1970), Szarski (1965), Lakshmikantham and Leela (1969) and the recent publication of Beesack (1975). Recently Snow (1972), Young (1973), Ghoshal and Masood (1974), Chandra and Davis (1976), Headley (1974), Bondge and Pachpatte (1979a, b, c) and Pachpatte (1975a, b; 1979a–c; 1980a–c) have established several new integral inequalities that have a wide range of applications in the theory of partial differential and integral equations. Our objective here is to establish n independent variable generalizations of the integral inequalities of Gollwitzer (1969), Langenhop (1960) and Pachpatte (1975a, b) which in turn are the further generalizations of the recent results of Bondge and Pachpatte (1979c). The inequalities established in this paper can be used in the analysis of some problems in the theory of integral equations involving n independent variables.

2. MAIN RESULTS

In this section we shall give some n independent variable generalizations of the integral inequalities of Gollwitzer (1969) and Langenhop (1960). We use the following notations throughout this paper.

Let $x = (x_1, ..., x_n)$ and $s = (s_1, ..., s_n)$ be points in R^n. We write $x < s$ (or $x \leqslant s$) if and only if $x_i < s_i$ (or $x_i \leqslant s_i$) for $1 \leqslant i \leqslant n$. Throughout this paper, Ω will denote an open set in R^n such that for any pair x, s of points of Ω with $x < s$ we have

$$D(x, s) = \{\xi \in R^n ; x \leqslant \xi \leqslant s\} \subset \Omega.$$

Throughout this paper

$$\int_x^s \dots d\xi \text{ denote the } n\text{-fold integral}$$

$$\int_{x_1}^{s_1} \dots \int_{x_n}^{s_n} \dots d\xi_n \dots d\xi_1;$$

and $$D_i = \frac{\partial}{\partial x_i}, \quad 1 \leqslant i \leqslant n.$$

A useful n independent variable generalization of Gollwitzer's inequality given in Lemma 2 (1969, p. 642) is embodied in the following theorem.

Theorem 1 — Let $\phi(x)$, $a(x)$, $b(x)$ be real-valued nonnegative continuous functions defined on Ω; let $u(x)$ be a positive real-valued continuous function defined on Ω, and suppose further that the inequality,

$$u(x) \geqslant \phi(x) - a(s) \int_x^s b(\xi)\, \phi(\xi)\, d(\xi) \qquad \dots(1)$$

is satisfied for $x \leqslant s$; $x, s \in \Omega$. Then

$$u(s) \geqslant \phi(x) \exp\left(-a(s) \int_0^s b(\xi)\, d\xi\right) \qquad \dots(2)$$

for $x \leqslant s$; $x, s \in \Omega$.

PROOF : Rewrite (1) as

$$\phi(x) \leqslant u(s) + a(s) \int_x^s b(\xi)\, \phi(\xi)\, d\xi. \qquad \dots(3)$$

For fixed $s \in \Omega$, we define for $x \leqslant s$,

$$r(x) = u(s) + a(s) \int_x^s b(\xi)\, \phi(\xi)\, d\xi \qquad \dots(4)$$

so $$r(s_1, x_2, \dots, x_n) = \dots = r(x_1, \dots, x_{n-1}, s_n) = u(s_1, \dots, s_n).$$

Then from (4) we have

$$D_1 r(x) = -a(s) \int_{x_2}^{s_2} \dots \int_{x_n}^{s_n} b(x_1, \xi_2, \dots, \xi_n)\, \phi(x_1, \xi_2, \dots, \xi_n)\, d\xi_n \dots d\xi_2 \qquad \dots(5)$$

and from (5) we have

$$D_1 D_2 r(x) = a(s) \int_{x_3}^{s_3} \dots \int_{x_n}^{s_n} b(x_1, x_2, \xi_3, \dots, \xi_n)\, \phi(x_1, x_2, \xi_3, \dots, \xi_n)\, d\xi_n \dots d\xi_3 \qquad \dots(6)$$

and in general we have

$$D_1 \dots D_k r(x) = (-1)^k \int_{x_{k+1}}^{s_{k+1}} \int_{x_n}^{s_n} b(x_1, \dots, x_k, \xi_{k+1}, \dots, \xi_n) \times \phi(x_1, \dots, x_k, \xi_{k+1}, \dots, \xi_n)\, d\xi_n \dots d\xi_{k+1} \quad \dots(7)$$

and continuing in this way, we obtain

$$D_1 \dots D_n r(x) = (-1)^n a(s)\, b(x)\, \phi(x). \quad \dots(8)$$

Now, we consider the following two cases.

Case I — If the order n of the derivatives in (8) is even, then from (8) we have

$$D_1 \dots D_n r(x) = a(s)\, b(x)\, \phi(x) \quad \dots(9)$$

which in view of (3) implies

$$D_1 \dots D_n r(x) \leqslant a(s)\, b(x)\, r(x) \quad \dots(10)$$

i.e.,

$$\frac{D_1 \dots D_n r(x)}{r(x)} \leqslant a(s)\, b(x). \quad \dots(11)$$

From (11) we observe that

$$\frac{r(x)\,[D_1 \dots D_n r(x)]}{r^2(x)} \leqslant a(s)\, b(x) + \frac{D_n r(x)\,[D_1 \dots D_{n-1} r(x)]}{r^2(x)}. \quad \dots(12)$$

For, by (4) we see that $D_n r(x)$ and $D_1 \dots D_{n-1} r(x)$ are both nonpositive which implies that $D_n r(x)\,[D_1 \dots D_{n-1} r(x)]$ is nonnegative and hence (12) is true. Now (12) is equivalent to

$$D_n \left(\frac{D_1 \dots D_{n-1} r(x)}{r(x)} \right) \leqslant a(s)\, b(x).$$

Now keeping $x_1, \dots, x_{n-1}$ fixed in the above inequality, set $x_n = \xi_n$ and then integrating with respect to ξ_n from x_n to s_n, we have

$$\frac{D_1 \dots D_{n-1} r(x)}{r(x)} \geqslant -a(s) \int_{x_n}^{s_n} b(x_1, \dots, x_{n-1}, \xi_n)\, d\xi_n. \quad \dots(13)$$

Again from (13), we observe that

$$\frac{r(x)\,[D_1 \dots D_{n-1} r(x)]}{r^2(x)} \geqslant -a(s) \int_{x_n}^{s_n} b(x_1, \dots, x_{n-1}, \xi_n)\, d\xi_n + \frac{D_{n-1} r(x)\,[D_1 \dots D_{n-2} r(x)]}{r^2(x)}. \quad \dots(14)$$

For, as before we see that $D_{n-1}r(x)$ is nonpositive and $D_1 \ldots D_{n-2}r(x)$ is nonnegative, which implies that $D_{n-1}r(x)\,[D_1 \ldots D_{n-2}r(x)]$ is nonpositive and hence (14) is true. But (14) is equivalent to

$$D_{n-1}\left(\frac{D_1 \ldots D_{n-2}r(x)}{r(x)}\right) \geqslant -a(s) \int_{x_n}^{s_n} b(x_1, \ldots, x_{n-1}, \xi_n)\, d\xi_n.$$

Now keeping $x_1, \ldots, x_{n-2}, x_n$ fixed in the above inequality, set $x_{n-1} = \xi_{n-1}$ and then integrating with respect to ξ_{n-1} from x_{n-1} to s_{n-1} we have,

$$\frac{D_1 \ldots D_{n-2}r(x)}{r(x)} \leqslant a(s) \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} b(x_1, \ldots, x_{n-2}, \xi_{n-1}, \xi_n)\, d\xi_n\, d\xi_{n-1}. \qquad \ldots(15)$$

Proceeding in this way, we finally obtain

$$\frac{D_1 r(x)}{r(x)} \geqslant -a(s) \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} b(x_1, \xi_2, \ldots, \xi_n)\, d\xi_n \ldots d\xi_2. \qquad \ldots(16)$$

Now keeping $x_2, \ldots, x_n$ fixed in (16), set $x_1 = \xi_1$ and then integrating with respect to ξ_1 from x_1 to s_1, we have

$$r(x) \leqslant u(s) \exp\left(a(s) \int_x^s b(\xi)\, d\xi\right). \qquad \ldots(17)$$

Substituting this bound on $r(x)$ in (3), we obtain the desired bound in (2).

Case II — If the order n of the derivatives in (8) is odd, then from (8), we have,

$$D_1 \ldots D_n r(x) = -a(s)\, b(x)\, \phi(x) \qquad \ldots(18)$$

which in view of (3) implies,

$$D_1 \ldots D_n r(x) \geqslant -a(s)\, b(x)\, r(x)$$

i.e.,

$$\frac{D_1 \ldots D_n r(x)}{r(x)} \geqslant -a(s)\, b(x). \qquad \ldots(19)$$

The rest of the proof for case II is precisely the same as for case I and the final inequality (17) remains unchanged since n is now odd.

Remark 1 : In Theorem 1, if we take $a(s) = M$, where $M > 0$ is a constant, then (2) reduces to

$$u(s) \geqslant \phi(x) \exp\left(-M \int_x^s b(\xi)\, d\xi\right).$$

As an application of Theorem 1, we next establish the following n independent variable generalization of Gollwitzer's inequality given in (1969, Theorem 1) for lower bound on unknown function.

Theorem 2 — Let $\phi(x)$, $a(x)$, $b(x)$ and $u(x)$ be as defined in Theorem 1, let $H(r)$ be a positive, continuous, strictly increasing, convex and submultiplicative function for $r > 0$, $H(0) = 0$; $\lim_{r \to \infty} H(r) = \infty$. Let $\alpha(s)$, $\beta(s)$ be positive continuous functions defined on Ω with $\alpha(s) + \beta(s) \equiv 1$. Suppose further that the inequality

$$u(s) \geqslant \phi(x) - a(s)\, H^{-1}\Big(\int_x^s b(\xi)\, H(\phi(\xi))\, d\xi \Big) \qquad \ldots(20)$$

is satisfied for $x \leqslant s$; $x, s \in \Omega$, then,

$$u(s) \geqslant \alpha(s)\, H^{-1}\left[\alpha^{-1}(s)\, H(\phi(x)) \exp\Big(-\beta(s)\, H(a(s)\, \beta^{-1}(s)) \Big) \int_x^s b(\xi)\, d\xi \right] \qquad \ldots(21)$$

for all $x \leqslant s$; $x, s \in \Omega$.

PROOF : The proof is identical to that given by Gollwitzer (1969). Rewrite (20) as

$$\phi(x) \leqslant \alpha(s)\, u(s)\, \alpha^{-1}(s) + \beta(s)\, a(s)\, \beta^{-1}(s)\, H^{-1}\Big(\int_x^s b(\xi)\, H(\phi(\xi))\, d\xi \Big).$$

Since H is convex, submultiplicative and monotonic, we have

$$\alpha(s)\, H(u(s)\, \alpha^{-1}(s)) \geqslant H(\phi(x)) - \beta(s)\, H(a(s)\, \beta^{-1}(s)) \Big(\int_x^s b(\xi)\, H(\phi(\xi))\, d\xi \Big).$$

Now an application of Theorem 1 yields the desired bound in (21).

Remark 2 : We note that in Theorem 2, if we take $H(u) = u$ then our Theorem 2 reduces to Theorem 1.

In Theorem 3 given below, we establish the following n independent variable generalization of the integral inequality established by Langenhop (1960).

Theorem 3 — Let $u(x)$, $a(x)$ and $b(x)$ be as defined in Theorem 1; let $W(r)$ be a positive, continuous, monotonic nondecreasing function for $r > 0$, $w(0) = 0$ and $w'(r)$ exists and is continuous, with $w'(r) \geqslant 0$ for $r \geqslant 0$; and suppose further that the inequality

$$u(s) \geqslant u(x) - a(s) \int_x^s b(\xi)\, W(u(\xi))\, d\xi \qquad \ldots(22)$$

is satisfied for $x \leqslant s$; $x, s \in \Omega$. Then, for $\Omega_1 \subset \Omega$,

$$u(s) \geqslant G^{-1}\left[G(u(x)) - a(s)\left(\int_x^s b(\xi)\, d\xi\right)\right] \quad \ldots(23)$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{W(s)}, \quad r > 0 \quad \ldots(24)$$

where r_0 is any fixed positive number; G^{-1} is the inverse function of G, and Ω_1 is such that

$$G(u(x)) - a(s)\left(\int_x^s b(\xi)\, d\xi\right) \in \mathrm{Dom}\,(G^{-1})$$

for all $x \leqslant s$; $x, s \in \Omega_1 \subset \Omega$.

PROOF: Rewrite (22) as

$$u(x) \leqslant u(s) + a(s)\left(\int_x^s b(\xi)\, W(u(\xi))\, d\xi\right). \quad \ldots(25)$$

For fixed $s \in \Omega$, we define for $x \leqslant s$, $x \in \Omega$,

$$r(x) = u(s) + a(s)\left(\int_x^s b(\xi)\, W(u(\xi))\, d\xi\right) \quad \ldots(26)$$

so $\qquad r(s_1, x_2, \ldots, x_n) = \ldots = r(x_1, \ldots, x_{n-1}, s_n) = u(s_1, \ldots, s_n).$

Then by the same argument as in the proof of Theorem 1 we obtain in general from (26) that

$$D_1 \ldots D_k r(x) = (-1)^k \int_{x_{k+1}}^{s_{k+1}} \ldots \int_{x_n}^{s_n} b(x_1, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n)$$
$$\times\, W(u(x_1, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n)\, d\xi_n \ldots d\xi_{k+1} \quad \ldots(27)$$

and continuing in this way, we obtain

$$D_1 \ldots D_n r(x) = (-1)^n a(s)\, b(x)\, W(u(x)). \quad \ldots(28)$$

We now consider the following two cases.

Case 1 — If the order n of the derivatives in (28) is even, then, from (28) we have

$$D_1 \ldots D_n r(x) = a(s)\, b(x)\, W(u(x)) \quad \ldots(29)$$

which in view of (25) implies

$$D_1 \dots D_n r(x) \leqslant a(s)\, b(x)\, W(r(x)) \qquad \dots(30)$$

i.e.,

$$\frac{D_1 \dots D_n r(x)}{W(r(x))} \leqslant a(s)\, b(x). \qquad \dots(31)$$

From (31) we observe that,

$$\frac{W(r(x))\,[D_1 \dots D_n r(x)]}{W^2(r(x))}$$
$$\leqslant a(s)\, b(x) + \frac{W'(r(x))\,.\,D_n(r(x))\,[D_1 \dots D_{n-1} r(x)]}{W^2(r(x))}. \qquad \dots(32)$$

For, by (34) we see that $D_n r(x)$ and $D_1 \dots D_{n-1} r(x)$ are nonpositive, which implies that $D_n r(x)\,[D_1 \dots D_{n-1} r(x)]$ is nonnegative and hence (32) is true. Now (32) is equivalent to

$$D_n \left(\frac{D_1 \dots D_{n-1} r(x)}{W(r(x))}\right) \leqslant a(s)\, b(x).$$

Now keeping $x_1, \dots, x_{n-1}$ fixed in the above inequality, set $x_n = \xi_n$ and then integrating with respect to ξ_n from x_n to s_n, we have

$$\frac{D_1 \dots D_{n-1} r(x)}{W(r(x))} \geqslant -a(s) \int_{x_n}^{s_n} b(x_1, \dots, x_{n-1}, \xi_n)\, d\xi_n. \qquad \dots(33)$$

Again from (33), we observe that,

$$\frac{W(r(x))\,[D_1 \dots D_{n-1} r(x)]}{W^2(r(x))} \geqslant -\, a(s) \int_{x_n}^{s_n} b(x_1, \dots, x_{n-1}, \xi_n)\, d\xi_n$$
$$+ \frac{W'(r(x)) D_{n-1}(r(x))\,[D_1 \dots D_{n-2} r(x)]}{W^2(r(x))}. \qquad \dots(34)$$

For, (34) shows that $D_{n-1} r(x)$ is nonpositive and $D_1 \dots D_{n-2} r(x)$ is nonnegative which implies that $D_{n-1} r(x)\,[D_1 \dots D_{n-2} r(x)]$ is nonpositive and hence (34) is true. Again, (34) is equivalent to

$$D_{n-1} \frac{D_1 \dots D_{n-2} r(x)}{W(r(x))} \geqslant -\, a(s) \int_{x_n}^{s_n} b(x_1, \dots, x_{n-1}, \xi_n)\, d\xi_n.$$

Now keeping $x_1, \dots, x_{n-2}, x_n$ fixed in the above inequality, set $x_{n-1} = \xi_{n-1}$ and then integrating with respect to ξ_{n-1} from x_{n-1} to s_{n-1}, we have

$$\frac{D_1 \ldots D_{n-2} r(x)}{W(r(x))} \leqslant a(s) \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} b(x_1, \ldots, x_{n-2}, \xi_{n-1}, \xi_n)\, d\xi_n\, d\xi_{n-1}.$$

Proceeding in this way we obtain

$$\frac{D_1 r(x)}{W(r(x))} \geqslant -a(s) \int_{x_2}^{s_2} \ldots \int_{x_n}^{s_n} b(x_1, \xi_2, \ldots, \xi_n)\, d\xi_n \ldots d\xi_2. \qquad \ldots(35)$$

From (24) and (35), we observe that,

$$D_1 G(r(x)) \geqslant -a(s) \int_{x_2}^{s_2} \ldots \int_{x_n}^{s_n} b(x_1, \xi_2, \ldots, \xi_n)\, d\xi_n \ldots d\xi_2. \qquad \ldots(36)$$

Now keeping $x_2, \ldots, x_n$ fixed in (36), set $x_1 = \xi_1$ and then integrating with respect to ξ_1 from x_1 to s_1, we have

$$G(r(x)) \leqslant G(u(s)) + a(s) \int_x^s b(\xi)\, d(\xi) \qquad \ldots(37)$$

which implies

$$G(u(s)) \geqslant G(u(x)) - a(s) \int_x^s b(\xi)\, d\xi. \qquad \ldots(38)$$

The desired bound in (23) follows from (38). The subdomain Ω_1 of Ω is obvious.

Case 2 — When n is odd; (28) becomes

$$D_1 \ldots D_n r(x) = -a(s)\, b(x)\, W(u(x))$$

and the proof proceeds exactly as in case 1, again leading to (38).

Remark 3 : We note that in Theorem 3, if we take $W(u) = u$, then (23) reduces to

$$u(s) \geqslant u(x) \exp\left(-a(s) \int_x^s b(\xi)\, d\xi\right)$$

and if we set, $W(u) = u^{\alpha}$, $0 < \alpha < 1$ then (23) reduces to

$$u(s) \geqslant \left[(u(x))^{\beta} - \beta a(s) \int_x^s b(\xi)\, d\xi \right]^{1/\beta}$$

where $\alpha + \beta = 1$.

3. Further Inequalities

In this section, we establish n independent variable generalizations of the integral inequalities recently established by Pachpatte (1975a, b) which can be used in some applications in the theory of hyperbolic partial integral and integrodifferential equations in n independent variables.

Our first result deals with the n independent variable generalization of the integral inequality recently established by Pachpatte (1975b, Theorem 1).

Theorem 4 — Let $\phi(x)$, $a(x)$, $b(x)$ and $c(x)$ be real-valued nonnegative continuous functions defined on Ω; let $u(s)$ be a positive real-valued continuous function defined on Ω; and suppose further that the inequality

$$u(s) \geqslant \phi(x) - a(s)\left[\int_x^s b(\xi)\,\phi(\xi)\,d\xi + \int_x^s b(\xi)\left(\int_\xi^s c(\zeta)\,\phi(\zeta)\,d\zeta\right)d\xi\right] \quad \ldots(39)$$

is satisfied for $x \leqslant s$; $x, s \in \Omega$. Then

$$u(s) \geqslant \phi(x)\left[1 + a(s)\left(\int_x^s b(\xi)\exp\left(\int_x^s [a(s)\,b(\zeta) + c(\zeta)]\,d\zeta\right)d\xi\right)\right]^{-1} \quad \ldots(40)$$

for $x \leqslant s$; $x, s \in \Omega$.

PROOF : Rewrite (39) as

$$\phi(x) \leqslant u(s) + a(s)\left[\int_x^s b(\xi)\,\phi(\xi)\,d\xi + \int_x^s b(\xi)\left(\int_\xi^s c(\zeta)\,\phi(\zeta)\,d\zeta\right)d\xi\right]. \quad \ldots(41)$$

For fixed $s \in \Omega$, we define for $x \leqslant s$; $x \in \Omega$,

$$r(x) = u(s) + a(s)\left[\int_x^s b(\xi)\,\phi(\xi)\,d\xi + \int_x^s b(\xi)\left(\int_\xi^s c(\zeta)\,\phi(\zeta)\,d\zeta\right)d\xi\right], \quad \ldots(42)$$

so $\quad r(s_1, x_2, \ldots, x_n) = \ldots = r(x_1, \ldots, x_{n-1}, s_n) = u(s_1, \ldots, s_n).$

Then, by following the same argument as in the proof of Theorem 1, we obtain in general, from (42) that

$$\begin{aligned} D_1 \ldots D_k r(x) = (-1)^k \Bigg[& \int_{x_{k+1}}^{s_{k+1}} \ldots \int_{x_n}^{s_n} b(x_1, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n) \\ & \times \phi(x_1, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n)\,d\xi_n \ldots d\xi_{k+1} \\ & + \int_{x_{k+1}}^{s_{k+1}} \ldots \int_{x_n}^{s_n} b(x_1, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n) \\ & \times \left(\int_{\xi_{k+1}}^{s_{k+1}} \ldots \int_{\xi_n}^{s_n} c(\xi_1, \ldots, \xi_k, \zeta_{k+1}, \ldots, \zeta_k) \right. \\ & \left. \times \phi(\xi_1, \ldots, \xi_k, \zeta_{k+1}, \ldots, \zeta_n)\,d\zeta_n \ldots d\zeta_{k+1}\right) d\xi_n \ldots d\xi_{k+1} \Bigg] \end{aligned} \quad \ldots(43)$$

and continuing in this way, we obtain

$$D_1 \ldots D_n r(x) = (-1)^n a(s)\, b(x) \left[\phi(x) + \int_x^s c(\xi)\, \phi(\xi)\, d\xi \right]. \qquad \ldots(44)$$

We now consider the following two cases.

Case I — If the order n of the derivatives in (44) is even, then from (44) and (41), we have

$$D_1 \ldots D_n r(x) \leqslant a(s)\, b(x) \left[r(x) + \int_x^s c(\xi)\, r(\xi)\, d\xi \right]. \qquad \ldots(45)$$

In (45) if we put

$$v(x) = r(x) + \int_x^s c(\xi)\, r(\xi)\, d\xi \qquad \ldots(46)$$

so $\qquad v(s_1, x_2, \ldots, x_n) = \ldots = v(x_1, \ldots, x_{n-1}, s_n) = r(s_1, \ldots, s_n) = u(s_1, \ldots, s_n).$

Then we have

$$D_1 \ldots D_n v(x) = D_1 \ldots D_n r(x) + c(x)\, r(x) \qquad \ldots(47)$$

since the order n of the derivative is even. Using (45) and the fact that $r(x) \leqslant v(x)$ in (47), we have

$$D_1 \ldots D_n v(x) \leqslant [a(s)\, b(x) + c(x)]\, v(x).$$

Now repeating the argument used in the proof of Theorem 1, we obtain the estimate

$$v(x) \leqslant u(s) \exp\left(\int_x^s [a(s)\, b(\xi) + c(\xi)]\, d\xi \right).$$

Now substitute this bound for $v(x)$ into (45) and carry out n successive integrations, using the fact that

$$D_1 \ldots D_k r(x_1, \ldots, \xi_{k+1}, \ldots, x_n) = 0$$

for $\xi_{k+1} = s_{k+1}$ by (43), to obtain

$$r(x) \leqslant u(s) \left[1 + a(s) \left(\int_x^s b(\xi) \exp\left(\int_\xi^s [a(s)\, b(\zeta) + c(\zeta)]\, d\zeta \right) d\xi \right) \right]. \qquad \ldots(48)$$

Substituting this bound on $r(x)$ in (41), we obtain the desired bound in (40).

Case II — When n is odd, (63) becomes

$$D_1 \ldots D_n r(x) = -a(s)\, b(x) \left[r(x) + \int_x^s c(\xi)\, r(\xi)\, d\xi \right] \qquad \ldots(49)$$

and the proof proceeds exactly as in case I, again leading to (48).

As an application of Theorem 4, we next establish the following n independent variable generalization of the integral inequality recently established by Pachpatte (1975b, Theorem 2).

Theorem 5 — Let $\phi(x)$, $a(x)$, $b(x)$ $c(x)$ and $u(x)$ be as defined in Theorem 4; let $H(r)$, $\alpha(s)$ and $\beta(s)$ be as defined in Theorem 2; and suppose further that the inequality

$$u(s) \geqslant \phi(x) - a(s)\, H^{-1}\Big[\int_x^s b(\xi)\, H(\phi(\xi))\, d\xi + \int_x^s b(\xi) \Big(\int_\xi^s c(\zeta)\, H(\phi(\zeta))\, d\zeta\Big)\, d\xi\Big] \quad \ldots(50)$$

is satisfied for $x \leqslant s$; $x, s \in \Omega$. Then

$$u(s) \geqslant \alpha(s)\, H^{-1}\Big[\alpha^{-1}(s)\, H(\phi(x))\, \{1 + \beta(s)\, H(a(s)\, \beta^{-1}(s)) \int_x^s b(\xi) \times \exp\Big(\int_\xi^s [\beta(s)\, H(a(s)\, \beta^{-1}(s))\, b(\zeta) + c(\zeta)]\, d\zeta\Big)\, d\xi\}^{-1}\Big] \quad \ldots(51)$$

for $x \leqslant s$; $x, s \in \Omega$.

Proof : Rewrite (50) as

$$\phi(x) \leqslant \alpha(s)\, u(s)\, \alpha^{-1}(s) + \beta(s)\, a(s)\, \beta^{-1}(s)\, H^{-1}\Big[\int_x^s b(\xi)\, H(\phi(\xi))\, d\xi + \int_x^s b(\xi)\Big(\int_\xi^s c(\zeta)\, H(\phi(\zeta))\, d\zeta\Big)\, d\xi\Big].$$

Since H is convex, submultiplicative and monotonic, we have

$$\alpha(s)\, H(u(s)\, \alpha^{-1}(s)) \geqslant H(\phi(x)) - \beta(s)\, H(a(s)\, \beta^{-1}(s))\Big[\int_x^s b(\xi)\, H(\phi(\xi))\, d\xi + \int_x^s b(\xi)\Big(\int_\xi^s c(\zeta)\, H(\phi(\zeta))\, d\zeta\Big)\, d\xi\Big].$$

Now an application of Theorem 4 yields the desired bound in (51).

Remark 4 : We note that, if $H(u) = u$, then our Theorem 5 reduces to Theorem 4.

Next we present the following n independent variable generalization of the integral inequality established by Pachpatte (1975a, Theorem 3).

Theorem 6 — Let $u(x)$, $a(x)$, $b(x)$ and $c(x)$ be as defined in Theorem 4. Let $G(r)$ be a positive, continuous, strictly increasing, subadditive and submultiplicative function for $r > 0$ with $G(0) = 0$, and let G^{-1} denote the inverse function of G. Suppose further that the inequality

$$u(s) \geqslant \phi(x) - a(s)\, G^{-1}\left[\int_x^s b(\xi)\, G(\phi(\xi))\, d\xi + \int_x^s b(\xi) \left(\int_\xi^s c(\zeta)\, G(\phi(\zeta))\, d\zeta \right) d\xi \right] \qquad \ldots(52)$$

is satisfied for $x \leqslant s$; $x, s \in \Omega$ where $\phi(x)$ is continuous and nonnegative on Ω. Then

$$u(s) \geqslant G^{-1}\left[G(\phi(x)) \{1 + G(a(s)) \left(\int_x^s b(\xi) \times \exp\left(\int_\xi^s [b(\zeta)\, G(a(s)) + c(\zeta)]\, d\zeta \right) d\xi \right)\}^{-1} \right] \qquad \ldots(53)$$

for $x \leqslant s$; $x, s \in \Omega$.

The proof of this theorem follows by the similar argument as in the proof of Theorem 3 given in Pachpatte (1975a) and using Theorem 4 with suitable modifications. We omit the details.

4. Some Applications

In this section, we present some applications of the inequalities established in this paper to obtain lower bounds on the solutions of integral equations in n independent variables. Each of these applications could be stated formally as a Theorem. This has not been done so as not to obscure the essential ideas with technical details.

Example 1 — As a first application we obtain the lower bound on the solution of a nonlinear integral equation of the form

$$u(x) = u(s) + \int_x^s F(\xi, u(\xi))\, d\xi, \quad x, s \in \Omega \qquad \ldots(54)$$

where all the functions involved in (54) are real-valued and defined on the respective domains of their definitions and

$$| F(x, u) | \leqslant b(x)\, W(| u |) \qquad \ldots(55)$$

where $b(x)$ and $W(r)$ are as defined in Theorem 3. Using (55) in (54) we have

$$| u(x) | \leqslant | u(s) | + \int_x^s b(\xi)\, W(| b(\xi) |)\, d\xi$$

i.e.,

$$| u(s) | \geqslant | u(x) | - \int_x^s b(\xi)\, W(| u(\xi) |)\, d\xi. \qquad \ldots(56)$$

Now assuming that $u(x)$, $(x < s;\ x, s \in \Omega)$ is positive and applying Theorem 3 we have

$$| u(s) | \geqslant G^{-1}\left[G(| u(x) |) - \int_x^s b(\xi)\, d\xi\right] \qquad ...(57)$$

where G and G^{-1} are as defined in Theorem 3. Thus the right-hand side of (57) gives us the lower bound on the solution $u(s)$ of (54).

Example 2 — As a second application, we establish the lower bound on the solution of a nonlinear integral equation of the form

$$u(x) = u(s) + \int_x^s F\left[\xi, u(\xi), \int_\xi^s k(\xi, \zeta, u(\zeta))\, d\zeta\right] d\xi,\ x, s \in \Omega \qquad ...(58)$$

where all the functions involved in (58) are real-valued and defined on the respective domains of their definitions and

$$| k(x, \xi, u) | \leqslant c(\xi) | u | \qquad ...(59)$$

$$| F[x, u, v] | \leqslant b(x) [| u | + | v |] \qquad ...(60)$$

where $b(x)$ and $c(x)$ are defined as in Theorem 4. Using (59) and (60) in (58) we have

$$| u(x) | \leqslant | u(s) | + \int_x^s b(\xi)\left[| u(\xi) | + \int_\xi^s c(\zeta) | u(\zeta) |\, d\zeta\right] d\xi$$

i.e.

$$| u(s) | \geqslant | u(x) | \left[1 + \int_x^s b(\xi) \exp\left(\int_\xi^s [b(\zeta) + c(\zeta)]\, d\zeta\right) d\xi\right]^{-1} \qquad ...(61)$$

which gives us the lower bound on the solution $u(s)$ of (58).

References

Beesack, P. R. (1975). Gronwall inequalities. *Carleton Mathematical Lecture Notes No.* 11.

Bondge, B. K., and Pachpatte, B. G. (1979a). On nonlinear integral inequalities of the Wendroff type. *J. Math. Anal. Applic.*, **70**.

———— (1979b). On Wendroff type integral inequalities in *n* independent variables. *Chinese J. Math.*, **7**, 37–45.

———— (1979c). On some fundamental integral inequalities in two independent variables. *J. Math. Anal. Applic.*, (to appear).

Chandra, J., and Davis, P. W. (1976). Linear generalization of Gronwall's inequality. *Proc. Am. math. Soc.*, **60**, 157–60.

Ghoshal, S. K., and Masood, M. A. (1974). Gronwall's vector inequality and its application to a class of non-self-adjoint linear and nonlinear hyperbolic partial differential equations. *J. Indian Math. Soc.*, **38**, 383–94.

Gollwitzer, H. E. (1969). A note on a functional inequality. *Proc. Am. math. Soc.*, **23**, 642–47.

Headley, V. B. (1974). A multidimentional nonlinear Gronwall inequality. *J. Math. Anal. Applic.*, **47**, 250–55.

Langenhop, C. E. (1960). Bounds on the norm of a solution of a general differential equation. *Proc. Am. math. Soc.*, **11**, 795–99.

Lakshmikantham, V., and Leela, S. (1969). Differential and Integral Inequalities, Vol. II. Academic Press, New York.

Pachpatte, B. G. (1975a). A note on integral inequalities of the Bellman-Bihari type. *J. Math. Anal. Applic.*, **49**, 295–301.

——— (1975b). A note on an integral inequality. *J. Math. Phys. Sci.*, **9**., 11–14.

——— (1979a). On some new integral and integrodifferential inequalities in two independent variables and their applications. *J. Diff. Eqns.*, **33**, 249–72.

——— (1979b). On Wendroff type partial integral inequalities. *Tamkang J. Math.*, **10**, 141–50.

——— (1979c). On some integrodifferential inequalities similar to Wendroff's inequality. *Ann. Sti. Univ. Al. I. Cuza, din Iasi*, **25**, 127–36.

——— (1980a). On some fundamental partial integral inequalities. *J. Math. Anal. Applic.*, **73**, (to apper).

——— (1980b). On some new integrodifferential inequalities of the Wendroff type. *J. Math. Anal. Applic.*, **73**, (to appear).

——— (1980c). On some new integral inequalities for nonself-adjoint hyperbolic partial integrodifferential equations. *J. Math. Anal. Applic.*, (to appear).

Snow, D. R. (1972). Gronwall's inequality for systems of partial differential equations in two independent variables. *Proc. Am. math. Soc.*, **33**, 46–54.

Szarski, J. (1965). Differential Inequalities. Warsaw.

Walter, W. (1970). Differential and Integral Inequalities. Heidelberg-Berlin, New York.

Young, E. C. (1973). Gronwall's inequality in n dependent variables. *Proc. Am. math. Soc.*, **41**, 241–44.