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PRODUCTS OF CLOSE-TO-STARLIKE AND CLOSE-TO-CONVEX FUNCTIONS

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In this paper the author obtains the orders of starlikeness, convexity and close-to-convexity of the products of functions from the classes of close-to-starlike and close-to-convex functions, by combining the technique used by Silverman (1975) with the results of Anh and Taun (1978).

1. INTRODUCTION

Let S denote the class of functions of the form $f(z) = z + \sum_{1}^{\infty} a_n z^n$ which are analytic and univalent in the unit disc $D = \{z/ \mid z \mid < 1\}$.

A function $f \in S$ is said to be starlike of order α, $(0 \leqslant \alpha \leqslant 1)$ denoted by $f \in S^*(\alpha)$ if $\text{Re}\left\{z \frac{f'}{f}\right\} > \alpha$, $(\mid z \mid < 1)$ and is said to be convex of order α denoted by $f \in K(\alpha)$ if $\text{Re}\left\{1 + z \frac{f''}{f'}\right\} > \alpha$, $(\mid z \mid < 1)$.

Kaplan (1952) calls a function $f(z)$ close-to-convex in $\mid z \mid < 1$ provided there is a function $g(z)$ analytic, univalent and convex in $\mid z \mid < 1$ for which $\text{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0$.

The problems of finding (i) the radius of starlikeness for the class $S_{\beta,\alpha}$ of functions $f(z) = z + a_2z^2 + \ldots$, $\mid z \mid < 1$ satisfying $\text{Re}\left\{\frac{f(z)}{F(z)}\right\} > \beta$, $0 \leqslant \beta < 1$ where $F(z) \in S^*(\alpha)$ and (ii) that of finding the radius of convexity for the class $K_{\beta,\alpha}$ of functions $f(z) = z + a_2z^2 + \ldots$ satisfying $\text{Re}\left\{\frac{f'(z)}{F'(z)}\right\} > \beta$ where $F(z) \in K(\alpha)$ have been discussed by Anh and Tuan (1978), the results obtained being sharp.

Silverman (1975) obtained the orders of starlikeness, convexity of products of starlike and convex functions of orders α and β respectively.

In this paper we discuss the more general problem of finding the orders of starlikeness and convexity of the products of functions from the classes $S_{\beta,\alpha}$ and $K_{\beta,\alpha}$.

2. Lemmas

The lemmas required for the proofs of our theorems are given in this section. These results are due to Anh and Tuan (1978).

Lemma 1 — Let $f(z) \in S$ be such that $\operatorname{Re}\left\{\frac{f(z)}{F(z)}\right\} > \beta,\ 0 \leqslant \beta < 1$ where $F(z) \in S^*(\alpha),\ 0 \leqslant \alpha < 1$. Then we have

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geqslant \frac{1+(2\alpha-1)r}{1+r} - \frac{2(1-\beta)r}{(1+(2\beta-1)r)(1+r)} = \mu_1(\beta, \alpha) \quad \text{for} \quad R_1 \leqslant R_2$$

$$\geqslant \frac{1+(2\alpha-1)r}{1+r} - \frac{\beta}{1-\beta} + \frac{1}{1-\beta}\left\{2\left(\frac{\beta-\beta(2\beta-1)r^2}{1-r^2}\right)^{1/2} - \frac{1-(2\beta-1)r^2}{1-r^2}\right\} = \mu_2(\beta, \alpha) \quad \text{for} \quad R_2 \leqslant R_1$$

where

$$R_1 = \left\{\frac{\beta-(2\beta-1)r^2}{1-r^2}\right\}^{1/2} \quad \text{and} \quad R_2 = \frac{1+(2\beta-1)r}{1+r}. \qquad \ldots(1)$$

The results are sharp.

Lemma 2 — Let $f(z) \in S$ be such that $\operatorname{Re}\left\{\frac{f'(z)}{F'(z)}\right\} > \beta,\ 0 \leqslant \beta < 1$ where $F(z) \in K(\alpha),\ 0 \leqslant \alpha < 1$ [the class of convex univalent functions of order α in the unit disc is denoted by $K(\alpha)$]. Then we have

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geqslant \mu_1(\beta, \alpha), \qquad R_1 \leqslant R_2$$

$$\geqslant \mu_2(\beta, \alpha), \qquad R_2 \leqslant R_1 \qquad \ldots(2)$$

where $\mu_1(\beta, \alpha)$ and $\mu_2(\beta, \alpha)$, R_1 and R_2 are defined as in Lemma 1. The results are sharp.

3. Orders of Starlikeness and Convexity Theorems

Theorem 1 — Suppose $f_i \in S_{\beta,\alpha}$, $i = 1, 2, \ldots n$ and $g_j \in K_{\beta,\alpha}$, $j = 1, \ldots m$. Let $h(z) = z \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right) a_i \prod_{j=1}^{m} (g_j'(z))\, b_j$ where $a_i, b_j \geqslant 0$. Set

$$\sum_{i=1}^{n} a_i = a \quad \text{and} \quad \sum_{j=1}^{m} b_j = b.$$

Then $h(z) \in S^*(1-(a+b)(1-\mu))$ where

$$\mu = \mu_1(\beta, \alpha) \quad \text{for} \quad R_1 \leqslant R_2$$

$$= \mu_2(\beta, \alpha) \quad \text{for} \quad R_2 \leqslant R_1.$$

This result is sharp.

PROOF OF THEOREM 1 : We have

$$\log h(z) = \log z + \sum_{i=1}^{n} a_i \{\log f_i(z) - \log z\} + \sum_{j=1}^{m} b_j \log g_j'(z).$$

Differentiating we get,

$$z\frac{h'(z)}{h(z)} = 1 + \sum_{i=1}^{n} a_i \left\{z\frac{f_i'(z)}{f_i(z)} - 1\right\} + \sum_{j=1}^{m} b_j \left\{\frac{zg_j''(z)}{g_j'(z)}\right\}$$

$$= 1 - a - b + \sum_{i=1}^{n} a_i \left\{z\frac{f_i'}{f_i}\right\} + \sum_{j=1}^{m} b_j \left\{1 + \frac{zg_j''(z)}{g_j'(z)}\right\}. \quad \ldots(3)$$

Using (1) and (2), (3) becomes on taking real parts

$$\text{Re}\,\frac{zh'(z)}{h(z)} \geqslant 1 - a - b + (a+b)\,\mu_1(\beta, \alpha) \quad \text{for} \quad R_1 \leqslant R_2$$

$$\geqslant 1 - a - b + (a+b)\,\mu_2(\beta, \alpha) \quad \text{for} \quad R_2 \leqslant R_1.$$

Thus

$$\text{Re}\,\frac{zh'(z)}{h(z)} \geqslant 1 - (a+b)(1-\mu)$$

where $\mu = \mu_1(\beta, \alpha)$ if $R_1 \leqslant R_2$

$= \mu_2(\beta, \alpha)$ if $R_2 \leqslant R_1$.

Sharpness follows by taking

$$\left.\begin{aligned} f_i(z) &= \frac{z}{(1+z)^{2-2\alpha}}\left\{\beta + (1-\beta)\frac{1-z}{1+z}\right\}, \quad i = 1, 2, \ldots, n \\ g_j(z) &= \int_0^z \left\{\beta + (1-\beta)\frac{1-\xi}{1+\xi}\right\}\frac{d\xi}{(1+\xi)^{2-2\alpha}}, \quad j = 1, 2, \ldots, m \end{aligned}\right\}\text{for } R_1 \leqslant R_2$$

and

$$f_i(z) = \frac{z}{(1+z)^{2-2\alpha}}\left\{\beta + \tfrac{1}{2}(1-\beta)\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right)\right\}$$

$$(i = 1, 2, \ldots n)$$

$$g_j(z) = \int_0^z \left[\beta + \tfrac{1}{2}(1-\beta)\left(\frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}} + \frac{1+\xi e^{i\theta}}{1-\xi e^{i\theta}}\right)\right]\frac{1}{(1+\xi)^{2-2\alpha}}\,d\xi$$

$$(j = 1, 2, \ldots m) \quad \text{for} \quad R_2 \leqslant R_1$$

where θ satisfies the equation

$$(2R_1 - a - \beta) - 2\cos\theta\,[(2R_1 - a - \beta)(1 + \beta) + (1 - \beta)^2]\,r + [2\beta(2R_1 - a - \beta)(1 + 2\cos^2\theta) + 4(1 - \beta)^2]\,r^2 - 2\cos\theta\,[(2R_1 - a - \beta)(3\beta - 1) + (1 - \beta)^2]\,r^3 + (2\beta - 1)\,r^4 = 0 \quad \ldots(4)$$

with $a = (1 - (2\beta - 1)r^2)/(1 - r^2)$.

Remarks : The results of Silverman (1975) follow by putting $\beta = 1$ and retaining α in (1). Then $f_i(z)$ become starlike of order α, $i = 1, 2, \ldots n$ and the corresponding $g_j(z)$ $j = 1, 2, \ldots m$ become convex of order β by putting $\beta = 1$ and replacing α by β in (2).

Then $\quad R_1 = R_2 = 1.$

Here
$$\text{Re}\,\frac{zh'(z)}{h(z)} \geqslant 1 - (a + b) + a\,\frac{1 + (2\alpha - 1)\,r}{1 + r} + b\,\frac{1 + (2\beta - 1)\,r}{1 + r}$$
$$\geqslant 1 - (a + b) + a\alpha + b\beta$$

as $r \to 1$, by the minimum principle for harmonic functions. Further the sharpness of these inequalities is given by the extreme functions :

$$f_i(z) = \frac{z}{(1 + z)^{2-2\alpha}} \quad \text{and} \quad g_j(z) = \int_0^z \frac{d\xi}{(1 + \xi)^{2-2\beta}}$$

as in Silverman (1975).

Theorem 2 — Let

$$H(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{a_i} \prod_{j=1}^{m} (g_j'(t))^{b_j}\,dt$$

where f_i, g_j, a_i, b_j and μ defined as in Theorem 1. Then

$$H(z) \in K\,[1 - (a + b)(1 - \mu)].$$

PROOF : $H(z) \in K(1 - (a + b)(1 - \mu))$ iff

$$zH'(z) = h(z) \in S^*\,[1 - (a + b)(1 - \mu)]$$

Now
$$zH'(z) = h(z) = z \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{a_i} \prod_{j=1}^{m} (g_j'(z))^{b_j}.$$

The result now follows from Theorem 1.

3. A Close-to-Convex Theorem

We will use the following result of Silverman (1975).

Lemma 3 — If $P(z)$ is analytic in $D = \{z/ \mid z \mid < 1\}$ with $P(0) = 1$ and Re $P(z) > \nu$. Then for $z = re^{i\theta}$ and $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$ we have

$$\nu(\theta_2 - \theta_1) \leqslant \int_{\theta_1}^{\theta_2} \text{Re } P(z)\, d\theta \leqslant 2\pi(1 - \nu) + \nu(\theta_2 - \theta_1).$$

Theorem 3 — Let $f_i \in S_{\beta,\alpha}$, $i = 1, 2, \ldots n$ and $g_j \in K_{\beta,\alpha}$, $j = 1, \ldots m$. Set

$$H(z) = \int_0^z \left\{ \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{a_i} \prod_{j=1}^{m} (g_j'(t))^{b_j} \right\} dt.$$

Put $$a = \sum_{i=1}^{n} a_i = \Sigma\, a_{i+} + \Sigma\, a_{i-} = a_+ + a_-$$

and $$b = \sum_{j=1}^{m} b_j = \Sigma\, b_{j+} + \Sigma\, b_{j-} = b_+ + b_-$$

where $\{a_{i+}\}$ and $\{b_{j+}\}$ are, respectively, the subsequences of $\{a_i\}$ and $\{b_j\}$ consisting of the positive terms and $\{a_{i-}\}$ and $\{b_{j-}\}$ are the subsequences consisting of the negative terms. Then $H(z)$ is close-to-convex, if

$$-\tfrac{1}{2} \leqslant (a_- + b_-)(1 - \mu_i(\beta, \alpha)) \leqslant (a_+ + b_+)(1 - \mu_i(\beta, \alpha)) \leqslant \tfrac{3}{2}$$

$i = 1, 2$ according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$

$\mu_i(\beta, \alpha)$, R_i, $i = 1, 2$ are defined as in Theorem 1.

These results are sharp.

Proof : For $H(z)$ to be close-to-convex, it is sufficient that

$$H'(z) \neq 0 \quad \text{and} \quad \int_{\theta_1}^{\theta_2} \text{Re} \left\{1 + z\frac{H''(z)}{H'(z)}\right\} d\theta \geqslant -\pi$$

for all θ_1, θ_2 such that $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$ and $0 < r = \mid z \mid < 1$. $H'(z) \neq 0$ evidently follows.

Here, we have

$$1 + z\frac{H''(z)}{H'(z)} = 1 - a - b + \sum_i a_i \frac{zf_i'}{f_i} + \sum_j b_j \left(\frac{1 + zg_j''}{g_j'}\right).$$

Taking real parts and integrating from θ_1 to θ_2 we obtain

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + z\frac{H''(z)}{H'(z)}\right) d\theta = (1-a-b)(\theta_2-\theta_1)$$

$$+ \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\sum_i a_{i+}\frac{zf'_i}{f_i} + \sum_j b_{j+}\left(1 + \frac{zg''_j}{g'_j}\right)\right\} d\theta$$

$$+ \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\sum_i a_{i-}\frac{zf'_i}{f_i} + \sum_j b_{j-}\left(1 + \frac{zg''_j}{g'_j}\right)\right\} d\theta.$$

Using Lemma 3 appropriately, we get,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zH''(z)}{H'(z)}\right) d\theta \geqslant (1-a-b)(\theta_2-\theta_1) + a_+\mu_l(\beta,\alpha)(\theta_2-\theta_1)$$

$$+ b_+\mu_l(\beta,\alpha)(\theta_2-\theta_1) + (a_- + b_-)\mu_l(\beta,\alpha)(\theta_2-\theta_1)$$

$$+ 2\pi\,[(a_-(1-\mu_l(\beta,\alpha)) + b_-(1-\mu_l(\beta,\alpha))]$$

$$= (\theta_2-\theta_1)\,[1-a-b+(a+b)\,\mu_l(\beta,\alpha)]$$

$$+ 2\pi(a_- + b_-)(1-\mu_l(\beta,\alpha))$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_1 \geqslant R_2$

$= L_l(\theta_2-\theta_1)$ (say), $l = 1, 2$ according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

$L_l(\theta_2-\theta_1)$ is a linear function of $(\theta_2-\theta_1)$ and assumes its minimum at either 0 or 2π depending of whether $(1-a-b)+(a+b)\,\mu_l(\beta,\alpha)$ is positive or negative $l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

Here $\quad L_l(0) = 2\pi(a_- + b_-)(1-\mu_l(\beta,\alpha))$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_1 \geqslant R_2$

$> -\pi$

when

$$(a_- + b_-)(1-\mu_l(\beta,\alpha)) \geqslant -\tfrac{1}{2} \qquad \ldots(5)$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

Also $\quad L_l(2\pi) = 2\pi\,[1-a-b+(a+b)\,\mu_l(\beta,\alpha)]$

$$+ 2\pi\,[(a_- + b_-)(1-\mu_l(\beta,\alpha))]$$

$$= 2\pi\,[1-(a_+ + b_+)(1-\mu_l(\beta,\alpha))]$$

$$> -\pi$$

when

$$(a_+ + b_+)(1 - \mu_l(\beta, \alpha)) \leqslant \tfrac{3}{2} \qquad \ldots(6)$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

Thus $H(z)$ will be close-to-covex whenever min $\{L(0), L(2\pi)\} \geqslant -\pi$. This is satisfied when both (5) and (6) are valid.

For $H(z)$ to be close-to-convex, therefore, we must have

$$-\tfrac{1}{2} \leqslant (a_- + b_-)(1 - \mu_l) \leqslant (a_+ + b_+)(1 - \mu_l) \leqslant \tfrac{3}{2} \qquad \ldots(7)$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$

Sharpness follows if we set

$$f_i = \frac{z}{(1+z)^{2-2\alpha}} \left\{\beta + (1-\beta)\frac{1-z}{1+z}\right\}, \quad g_j = \int_0^z \left[\beta + (1-\beta)\frac{1-\xi}{1+\xi}\right]\frac{d\xi}{(1+\xi)^{2-2\alpha}}$$

$$(i = 1, 2, \ldots, n) \qquad (j = 1, 2, \ldots, m)$$

(if $R_1 \leqslant R_2$)

and

$$f_i = \frac{z}{(1+z)^{2-2\alpha}} \left\{\beta + \tfrac{1}{2}(1-\beta)\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right)\right\}$$

$$(i = 1, 2, \ldots, n)$$

$$g_j = \int_0^z \left\{\beta + \tfrac{1}{2}(1-\beta)\left(\frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}} + \frac{1+\xi e^{i\theta}}{1-\xi e^{i\theta}}\right)\right\} \frac{d\xi}{(1+\xi)^{2-2\alpha}}$$

$$(j = 1, 2, \ldots, m)$$

where θ satisfies (4).

Remarks: The result of Silverman (1975) is obtained as a particular case by putting $\beta = 1$ and retaining α in (1) and putting $\beta = 1$ and replacing α by β in (2). Then $f_i(z) \in S^*(\alpha)$, $i = 1, 2, \ldots, n$ and $g_j(z) \in K(\beta)$, $j = 1, 2, \ldots, m$. Also

$$R_1 = R_2 = 1.$$

Here (7) becomes

$$-\tfrac{1}{2} \leqslant (1-\alpha)a_- + (1-\beta)b_- \leqslant (1-\alpha)a_+ + (1-\beta)b_+ \leqslant \tfrac{3}{2}.$$

Sharpness is obtained if we set $f_i = \dfrac{z}{(1+z)^{2-2\alpha}}$, $i = 1, \ldots, n$

and $$g_j(z) = \int_0^z \frac{d\xi}{(1+\xi)^{2-2\alpha}}, \quad j = 1, \ldots, m.$$

4. Related Classes

Theorem 4 — Suppose

$$f_i \in S_{\lambda,\alpha_i},\ i = 1, 2, \ldots, n \text{ and } g_j \in K_{\lambda,\beta_j},\ j = 1, 2, \ldots m.$$

Let
$$h(z) = z \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^a \prod_{j=1}^{m} (g_j'(z))^b, \quad a, b > 0$$

Put
$$n\alpha^* = \sum_{i=1}^{n} \alpha_i \text{ and } m\beta^* = \sum_{j=1}^{m} \beta_j.$$

Then
$$h(z) \in S^* \{1 - na(1 - \mu_l(\lambda, \alpha^*)) - mb(1 - \mu_l(\lambda, \beta^*))\}$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$. The result is sharp.

Proof: Logarithmic derivation gives

$$\frac{zh'(z)}{h(z)} = 1 + a \sum_{i=1}^{n} \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + b \sum_{j=1}^{m} \frac{zg_j''(z)}{g_j'(z)}$$

$$= 1 - (na + mb) + a \sum_{i=1}^{n} \frac{zf_i'(z)}{f_i(z)} + b\left(1 + \sum_{j=1}^{m} \frac{zg_j''(z)}{g_j'(z)}\right).$$

Therefore

$$\text{Re}\, \frac{zh'(z)}{h(z)} \geqslant 1 - (na + mb) + na\mu_l(\lambda, \alpha^*) + mb\mu_l(\lambda, \beta^*)$$

$$= 1 - na(1 - \mu_l(\lambda, \alpha^*)) - mb(1 - \mu_l(\lambda, \beta^*))$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

Sharpness is obtained by taking

$$\left.\begin{aligned} f_i(z) &= \frac{z}{(1+z)^{2-2\alpha_i}} \left\{\lambda + (1-\lambda)\frac{1-z}{1+z}\right\},\ i = 1, \ldots, n \\ g_j(z) &= \int_0^z \left\{\lambda + (1-\lambda)\frac{1-\xi}{1+\xi}\right\} \frac{d\xi}{(1+\xi)^{2-2\beta_j}},\ j = 1, \ldots, m \end{aligned}\right\} \text{ when } R_1 \leqslant R_2$$

and

$$f_i(z) = \frac{z}{(1+z)^{2-2\alpha_i}} \left\{\lambda + \tfrac{1}{2}(1-\lambda)\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{-i\theta}}\right)\right\},$$

$$i = 1, 2, \ldots, n$$

$$g_j(z) = \int_0^z \left\{\lambda + \tfrac{1}{2}(1-\lambda)\left(\frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}} + \frac{1+\xi e^{i\theta}}{1-\xi e^{i\theta}}\right)\right\} \frac{d\xi}{(1+\xi)^{2-2\beta_j}},$$

$$j = 1, 2, \ldots, m$$

when $R_2 \leqslant R_1$.

Here θ satisfies eqn. (4).

Remarks : As a particular case when $\lambda = 1$,

$f_i \in S^*(\alpha_i)$ and $g_j \in K(\beta_j)$ and $h(z) \in S^*(1 - an(1-\alpha^*) - bm(1-\beta^*))$ since in this case $R_1 = R_2$ and using the minimum principle for harmonic functions.

This is Silverman's result (1975) with sharpness given by (on putting $\lambda = 1$)

$$f_i(z) = \frac{z}{(1+z)^{2-2\alpha_i}}, \quad i = 1, 2, \ldots, n$$

$$g_j(z) = \int_0^z \frac{d\xi}{(1+\xi)^{2-2\beta_j}}, \quad j = 1, \ldots, m$$

Order of Convexity

Theorem 5 — If $f_i \in S_{\lambda,\alpha_i}$, $i = 1, 2, \ldots, n$ and $g_j \in K_{\lambda,\beta_j}$, $j = 1, 2, \ldots, m$ then

$$H(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^a \prod_{j=1}^m (g_j'(t))^b \, dt$$

$$\in K[1 - na(1 - \mu_l(\lambda, \alpha^*)) - mb(1 - \mu_l(\lambda, \beta^*))]$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$. The result is sharp.

PROOF : The result follows from Theorem 4 on using the fact that H is convex of order ρ iff zH' is starlike of order ρ.

Order of Close-to-Convexity

Theorem 6 — Under the same conditions of Theorem 5, allowing a and b now to be any real numbers, $H(z)$ is close-to-convex if

$$-\tfrac{1}{2} \leqslant a\left(m - \sum_{i=1}^n \mu_l(\lambda, \alpha_i)\right) + b\left(m - \sum_{j=1}^m \mu_l(\lambda, \beta_j)\right) \leqslant \tfrac{3}{2} \text{ whenever } ab > 0,$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$ or

$$\left.\begin{aligned} -\tfrac{1}{2} &\leqslant a\left(n - \sum_{i=1}^n \mu_l(\lambda, \alpha_i)\right) \leqslant \tfrac{3}{2} \\ -\tfrac{1}{2} &\leqslant b\left(m - \sum_{j=1}^m \mu_l(\lambda, \beta_j)\right) \leqslant \tfrac{3}{2} \end{aligned}\right\} \text{ whenever } ab < 0$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

The results are sharp.

PROOF : We have

$$\left(1 + z\frac{H''}{H'}\right) = 1 - na - mb + a\sum_{i=1}^{n} z\frac{f'_i}{f_i} + b\sum_{j=1}^{m}\left(1 + z\frac{g''_j}{g'_j}\right)$$

Using Lemma 3, we get,

$$(\theta_2 - \theta_1)\sum_{i=1}^{n} \mu_l(\lambda, \alpha_i) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\sum_{i=1}^{n} \frac{zf'_i}{f_i}\right) d\theta$$

$$\leqslant 2\pi\,[n - \sum_{i=1}^{n} \mu_l(\lambda, \alpha_i)] + (\theta_2 - \theta_1)\sum_{i=1}^{n} \mu_l(\lambda, \alpha_i)$$

and

$$(\theta_2 - \theta_1)\sum_{j=1}^{m} \mu_l(\lambda, \beta_j) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\sum_{j=1}^{m}\left(1 + \frac{zg''_j}{g'_j}\right) d\theta$$

$$\leqslant 2\pi\,[m - \sum_{j=1}^{m} \mu_l(\lambda, \beta_j)] + (\theta_2 - \theta_1)\sum_{j=1}^{m} \mu_l(\lambda, \beta_j)$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$. Using the inequalities either on the left or right of the above two expressions according as a and b are positive or negative and minimizing

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zH''}{H'}\right) d\theta \quad \text{over all} \quad 0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$$

as in Theorem 3 above, we first determine when the appropriate minimums are $\geqslant -\pi$.

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zH''(z)}{H'(z)}\right\} d\theta \geqslant -\pi$$

iff $$-\tfrac{1}{2} \leqslant a(n - \sum_{i=1}^{n} \mu_l(\lambda, \alpha_i)) + b(m - \sum_{j=1}^{m} \mu_l(\lambda, \beta_j))_j \leqslant \tfrac{3}{2} \text{ whenever } ab > 0,$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$ or

$$\left.\begin{aligned} -\tfrac{1}{2} &\leqslant a\,[n - \sum_{i=1}^{n} \mu_l(\lambda, \alpha_i)] \leqslant \tfrac{3}{2} \\ -\tfrac{1}{2} &\leqslant b\,[m - \sum_{j=1}^{m} \mu_l(\lambda, \beta_j)] \leqslant \tfrac{3}{2} \end{aligned}\right\} \quad \text{whenever} \quad ab < 0$$

$l = 1$ or 2 according as $R_1 \leqslant R_2$ or $R_2 \leqslant R_1$.

Sharpness is found by using the same functions as in Theorem 5 above.

Remarks : On taking $\lambda = 1$ we get, $R_1 = R_2$, $f_i \in S^*(\alpha_i)$, $(i = 1, 2 \ldots n)$ $g_j \in K(\beta_j)$, $(j = 1, 2, \ldots m)$. Proceeding to the limit as $r \to 1$, we get

$$\left.\begin{array}{l} -\frac{1}{2} \leqslant na\,[1 - \alpha^*] \leqslant \frac{3}{2} \\ -\frac{1}{2} \leqslant mb\,[1 - \beta^*] \leqslant \frac{3}{2} \end{array}\right\} \text{ whenever } ab < 0$$

and $$-\tfrac{1}{2} \leqslant an(1 - \alpha^*) + mb(1 - \beta^*) \leqslant \tfrac{3}{2} \quad \text{whenever } ab > 0.$$

Sharpness is found from

$$f_i = \frac{z}{(1 - z)^{2(1-\alpha_i)}} \quad (i = 1, 2, \ldots n)$$

and $$g_j = \int_0^z \frac{dt}{(1 - z)^{2(1-\beta_j)}} \quad (j = 1, 2, \ldots m).$$

This is strictly in accordance with Silverman (1975).

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