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REMARKS ON CERTAIN GENERALIZED EXPANSIONS

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Using an identity of Gould (1961) two general expansions are obtained which contain as special case number of expansion formulas of Cohen (1976).

§1. Using some differential operators, Cohen (1976) has proved the expansion :

$$\sum_{k=0}^{\infty} \frac{\beta}{\beta+k} \frac{(lk+1)^k}{k!} (-xe^{lx})^k = e^{-x}\, {}_1F_1[1;\, 1+\beta;\, x(1-\beta l)] \qquad \ldots(1.1)$$

which contains, as special cases, the following two expansions given in Polya and Szego (1964, problems 210 and 214) :

$$\sum_{n=0}^{\infty} \frac{(ln+1)^{n-1}}{n!} \omega^n = e^{-x}, \quad \omega = -xe^{xl} \qquad \ldots(1.2)$$

$$\sum_{n=0}^{\infty} \frac{(ln+1)^n}{n!} \omega^n = \frac{e^{-x}}{1+lx}. \qquad \ldots(1.3)$$

Zeitlin (1970) [see also Carlitz (1968) and Srivastava (1977, p. 330)] observed the known fact that the following consequences of Lagrange's expansion theorem :

$$\sum_{p=0}^{\infty} \frac{\Gamma(\alpha+bp)}{p!\,\Gamma(\alpha+bp-p)} \cdot \frac{x^p}{(1+x)^{bp}} = \frac{(1+x)^\alpha}{1+x(1-b)} \qquad \ldots(1.4)$$

$$\sum_{p=0}^{\infty} \frac{(\alpha)_{bp}}{p!\,(\alpha+1)_{bp-p}} \cdot \frac{x^p}{(1+x)^{bp}} = (1+x)^\alpha \qquad \ldots(1.5)$$

[incorporating (1.2) and (1.3) as special cases and employed by Cohen (1976) for obtaining his expansions] are equivalent to each other. In fact, the known expansions (1.2) – (1.5) are all special cases of the following identity of Gould (1961) [see also Verma (1974) and Zeitlin (1970)] :

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$$\sum_{n=0}^{\infty} \frac{\omega}{\omega + bn} \binom{a + bn}{n} v^n$$

$$= (1 + x)^a \sum_{n=0}^{\infty} (-)^n \binom{a - \omega}{n} \binom{\omega b^{-1} + n}{n}^{-1} \left(\frac{x}{1 + x}\right)^n \qquad \ldots(1.6)$$

where $v(1 + x)^b = x$. Indeed it is easy to check that (1.1) is also a special case of (1.6) [in (1.6) replacing b, x and w by al, $-z/a$ and $a\beta l$, respectively, and letting $a \to \infty$]. From these observations it is easy to see that not only (2.1a) and (2.2b); (2.4c) and (2.5b) of Cohen (1976) are equivalent, but also his Theorems (4.1a) and (4.2b) are equivalent. It is also apparent that Theorems (2.1a), (2.2b) and (3.1a) of Cohen (1976) are included in his Theorems (4.1a), (4.2b) and (5.1a), respectively.

In this note we begin in section 2 by giving an alternative proof of (1.1) by using Lagrange's expansion theorem (Whittaker and Watson 1958). In section 3 and 4 we obtain expansion theorems that incorporate the expansion theorems of Cohen (1976) as special cases.

§2. By Lagrange's expansion theorem (Whittaker and Watson 1958), we have

$$e^{-\beta lz}\, {}_1F_1[\beta; 1 + \beta; z(\beta l - 1)]$$

$$= \sum_{n=0}^{\infty} D^n\{(lz + 1)\, e^{-lz(\beta+n)}\, {}_1F_1[\beta; 1 + \beta; z(\beta l - 1)]\}\big|_{z=0} \frac{(ze^{lz})^n}{n!}$$

$$= \sum_{n,j=0}^{\infty} \frac{\beta}{\beta + j} \frac{(\beta l - 1)^j}{j!} \sum_{k=0}^{n} \binom{n}{k} [\{lD^k z^{j+1} + D^k z^j\} (-\beta - n)^{n-k}$$

$$\times\, e^{-z(\beta+n)l}]_{z=0} \frac{(ze^{lz})^n}{n!}$$

$$= \sum_{n=0}^{\infty} (\beta + n)^n \sum_{j=0}^{\infty} \frac{\beta}{\beta + j} \frac{(\beta l - 1)^j}{j!} \left\{\frac{(-n)_{j+1}}{(\beta + n)^{j+1}} + \frac{(-n)_j}{(\beta + n)^j}\right\} \frac{(ze^{lz})^n}{n!}$$

$$= \beta \sum_{n=0}^{\infty} \frac{(-lze^{lz})^n}{n!} (\beta + n)^{n-1} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{\beta l - 1}{\beta l + nl}\right)^j$$

$$= \sum_{n=0}^{\infty} \frac{\beta}{\beta + n} \frac{(1 + ln)^n}{n!} (-ze^{lz})^n. \qquad \ldots(2.1)$$

Transforming the ${}_1F_1$ in the left-hand side of (2.1) by Kummer's transformation (Rainville 1960, p. 125), we get (1.1).

§3. We begin this section by proving the following expansion formula which contains a number of theorems of Cohen (1976) as special cases :

$$\sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{(1+\alpha)_{bn}}{(1+\alpha)_{bn-n}} F_n(x)\, G_n(y) = \sum_{k,\, n=0}^{\infty} \binom{\alpha-\beta}{n}$$

$$\times \binom{(\beta+bn+bsk+lk)/b}{n}^{-1} \{e_k\} \frac{x^k}{k!} \Big[\{f_{n+sk}\}\, y^{n+sk} + n\Big(1+\frac{l}{s}\Big)$$

$$\times \sum_{p=0}^{\infty} \frac{\left(n+\frac{ln}{s}+\frac{lp}{s}+\frac{l}{s}+1\right)_p}{(p+1)!}\, y^{n+sk+p+1}\, \{f_{n+sk+p+1}\}\Big] \qquad \ldots(3.1)$$

where $$G_n(y) = \sum_{p=0}^{\infty} \frac{\left(bn+\alpha+\frac{ln}{s}\right)_{(ps+lp)/s}}{p!\left(bn+\alpha+1+\frac{ln}{s}\right)_{(lp)/s}}\, y^{n+p}\, \{f_{n+p}\}$$

and $$F_n(x) = \sum_{k=0}^{[n/s]} \frac{\beta+bsk+lk}{\beta+bn+lk} \frac{(-n)_{sk}\,(1+\alpha+bn)_{lk}}{k!\,(1+\alpha+bn-n)_{lk+sk}}\, \{e_k\}\, x^k.$$

Proof of (3.1) — In (1.6) replacing a, w and x by $\alpha+\beta sk+lk$, $\beta+bsk+lk$ and $-z$ respectively, multiplying both sides by $\{e_k\}\, x^k z^{sk}/k!(1-z)^{bsz}$, and summing from $k=0$ to ∞, we get

$$\sum_{k,\, n=0}^{\infty} \frac{\beta+bsk+lk}{\beta+bsk+lk+bn} \binom{\alpha+bn+bsk+lk}{n} \frac{\{e_k\}}{k!} \frac{(-x)^n\, z^{n+sk}}{(1-z)^{bn+bsk}}$$

$$= \sum_{k=0}^{\infty} \frac{\{e_k\}}{k!} x^k z^{sk} (1-z)^{\alpha+lk} \sum_{n=0}^{\infty} \binom{\alpha-\beta}{n} \binom{\frac{\beta+bnsk+lk}{b}+n}{n}^{-1}$$

$$\times \left(\frac{z}{1-z}\right)^n.$$

Now in view of the transformation

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/s]} f(n,k) = \sum_{n,\, k=0}^{\infty} f(n+sk,\, k) \qquad \ldots(3.2)$$

the above expression may be rewritten as

$$\sum_{n=0}^{\infty} \frac{(1+\alpha)_{bn}}{n!\,(1+\alpha)_{bn-n}} \frac{(-z)^n}{(1-z)^{bn}} F_n(x) = (1-z)^{\alpha} \sum_{n,\,k=0}^{\infty} \binom{\alpha-\beta}{n}$$

$$\times \binom{\frac{\beta + bsk + lk}{b} + n}{n}^{-1} \frac{\{e_k\}}{k!} \frac{x^k z^{n+sk}}{(1-z)^{n-lk}}. \qquad \ldots(3.3)$$

If we write $z(1-z)^{l/s} = y$ and use Lagrange's expansion theorem, we can write

$$z^{n+sk}(1-z)^{-n+lk}$$

$$= \sum_{j=0}^{\infty} \frac{y^j}{j!} \left[D^j \left\{ z^{n+sk}(1-z)^{-n+lk-1-(lj/s)} \left(1 - z - \frac{lz}{s}\right)\right\}\right]_{z=0}$$

$$= \sum_{j=n+sk}^{\infty} \frac{y^j}{(j-n-sk)!} (-lk + lj + j - sk)_{n-j+sk+2} - \left(1 + \frac{l}{s}\right)$$

$$\times \sum_{j=n+sk+1}^{\infty} \frac{(-lk + (lj/s) + j - sk - 1)_{n-j+sk+1}\, y^j}{(j-n-sk-1)!}$$

$$= y^{sk+n} + n\left(1 + \frac{l}{s}\right) y^{n+sk+1} + y^{n+sk} \sum_{p=2}^{\infty} \frac{y^p}{p!} \left(n + \frac{ln}{s} + \frac{lp}{s} + 1\right)_p$$

$$- \left(1 + \frac{l}{s}\right) y^{n+sk+1} \sum_{p=1}^{\infty} \frac{y^p}{p!} \left(n + 1 + \frac{ln}{s} + \frac{lp}{s} + \frac{l}{s}\right)_p$$

$$= y^{sk+n} + n\left(1 + \frac{l}{s}\right) y^{n+sk+1} \left[1 + \sum_{p=1}^{\infty} \frac{y^p}{(p+1)!}\right.$$

$$\left. \times \left(n + 1 + \frac{ln}{s} + \frac{l}{s} + \frac{lp}{s}\right)_p\right.$$

$$= y^{sk+n} + n\left(1 + \frac{l}{s}\right) y^{n+sk+1} \sum_{p=0}^{\infty} \frac{y^p}{(p+1)!}$$

$$\times \left(n + 1 + \frac{ln}{s} + \frac{l}{s} + \frac{lp}{s}\right)_p. \qquad \ldots(3.4)$$

Using (3.4) and (1.5) [with α, b and x replaced by $1 + ls^{-1}$, $\alpha + bn + lns^{-1}$ and $z/(1-z)$ respectively] in (3.3) and taking transforms with respect to y on both the sides, we get (3.1).

For $\beta = \alpha$, (3.3) reduces to (5.1a) of Cohen. Whereas Theorem (5.1a) of Cohen (1976) on replacing $m, y, x, \{f_n\}$ and $\{e_k\}$ by $l\alpha, z/\alpha, x\alpha^s, (-)^n\{d_n\}$ and $\frac{(-)^{sk}\{e_k\}}{1+msk}$ respectively and then letting $\alpha \to \infty$, will yield Theorem (3.1a) of Cohen (1976). On the other hand, for $\beta = \alpha$ (3.1) reduces to Theorem (4.2b) of Cohen and in (3.1) letting $\beta \to \infty$, we get Theorem (4.1a) of Cohen (on replacing α by $\alpha - 1$). Next in Theorem (4.1a) of Cohen replacing z, m, l and $\{e_k\}$ by $-z/\alpha$, $l\alpha$, 0 and $(-)^{sk}\alpha^{sk}\{e_k\}$ respectively and then letting $\alpha \to \infty$, we obtain Theorem (2.1a) of Cohen (1976). Moreover, in Theorem (4.2b) of Cohen replacing m, l, z and $\{e_k\}$ by $l\alpha, 0, -z/\alpha$ and $(-)^{sk}\alpha^{sk}\{e_k\}/(1+lsk)$ respectively and then letting $\alpha \to \infty$, we get Theorem (2.2b) of Cohen.

§4. In this section we prove the following expansion which contains the remaining theorems of Cohen (1976) as special cases :

$$\sum_{n,p=0}^{\infty} \frac{(-\alpha - mn - (ln/s))_{-lp}}{(-\alpha - mn + 1 - (ln/s))_{-lp-p}} \frac{y^{n+sp}}{n!p!} \{d_{n+sp}\}$$

$$\times \sum_{j=0}^{[n/s]} \frac{(-n)_{sj}\,(-)^{sj}\,\alpha}{\beta + mn - lj + (ln/s)} \binom{\alpha + mn - lj + (ln/s)}{j} x^{n-sj}\{c_{n-sj}\}$$

$$= \sum_{n=0}^{\infty} \frac{\alpha\{c_n\}\,x^n}{(\beta + mn + (ln/s))\,n!} \sum_{j=0}^{\infty} (-)^j \binom{\alpha - \beta}{j}$$

$$\times \binom{\frac{\beta + mn + (ln/s)}{b} + j}{j}^{-1} [y^{n+sj}\{d_{n+sj}\} - j(1+l)$$

$$\times \sum_{p=0}^{\infty} (-)^p \frac{(1 + l + j + lp + lj)_p}{(p+1)!} y^{n+sp}\{d_{n+sp}\}]. \qquad \ldots(4.1)$$

Proof of (4.1) — In (1.6) replacing a and w by $mn + \alpha + (ln/s)$ and $mn + \beta + (ln/s)$ respectively, summing from n 0 to ∞ after multiplying both sides by $\alpha x^n z^{n/s}\{c_n\}/(\beta + mn + (ln/s))\,n!\,(1+z)^{bn/s}$ and transforming the expression on the left-hand side by (3.2), we get

$$\sum_{n=0}^{\infty} \frac{z^{n/s}}{n!\,(1+z)^{bn/s}} \sum_{j=0}^{[n/s]} \binom{\alpha + mn - msj + (ln/s) - lj + bj}{j}$$

$$\times (-)^{sj} \frac{\alpha(-n)_{sj}\,\{c_{n-sj}\}\,x^{n-sj}}{\beta + mn - msj + (ln/s) - lj + bn} =$$

(*equation continued on p.* 348)

$$= \sum_{n=0}^{\infty} \frac{\alpha x^n z^{n/s}}{n!\,(\beta + mn + (ln/s))} (1 + z)^{(\alpha s + mns + ln - bn)/s} \{c_n\} \sum_{j=0}^{\infty} \binom{\alpha - \beta}{j}$$

$$\times \binom{\frac{\beta + mn + (ln/s)}{b} + j}{j}^{-1} \left(-\frac{z}{1+z}\right)^j. \qquad \ldots(4.2)$$

But in view of (1.5), we have

$$z^{n/s}(1 + z)^{-\alpha - mn} = \sum_{p=0}^{\infty} \frac{y^{n+sp}}{p!} \frac{(-\alpha - mn - (ln/s))_{-lp}}{(-\alpha - mn + 1 - (ln/s))_{-p-lp}}, \qquad \ldots(4.3)$$

where $y = z^{1/s}(1 + z)^{l/z}$. Expanding $z^{j+(n/s)}(1 + z)^{(ln/s)-j}$ in powers of y by Lagrange's theorem, we shall have

$$z^{j+(n/s)}(1 + z)^{(ln/s)-j} = y^{n+sj} - j(1 + l) \sum_{p=0}^{\infty} (-)^p \frac{y^{n+sp}}{(p+1)!}$$
$$\times (1 + l + j + lp + lj)_p. \qquad \ldots(4.4)$$

Using (4.3) and (4.4) in (4.2) with $b = ms$ and taking transform with respect to y, we get (4.1).

For $\alpha = \beta$, (4.2) reduces to an expansion which is analogous to Theorem 3 of Cohen (1976). On the other hand, in (4.2) setting $\beta = \alpha$ and replacing b, m, z and $\{c_n\}$ by ms, $l\alpha$, z/α and $z^{n/s}\{c_n\}$ respectively we get Theorem (2.5d) of Cohen on letting $\alpha \to \infty$. However, multiplying (4.2) by β and then letting $\beta \to \infty$, we get a result from which Theorem (2.4c) of Cohen follows as a special case. Lastly, (4.1) on setting $\beta = \alpha$ and then replacing m, x, y by $l\alpha$, $x\alpha^{1/s}$, $y\alpha^{-1/s}$ respectively yields Theorem (3.4c) of Cohen on letting $\alpha \to \infty$.

Indeed, some of the special cases cited by Cohen for instance his (2.18), (2.19), (3.13), (3.15) and (4.11) are in fact deducible from an already known result Verma (1965, 1974).

It may be worthwhile to mention as a closing remark that starting with the Euler's transformation for ${}_2F_1$ (Rainville 1960, p. 60).

$${}_2F_1\left[\begin{matrix} c, \lambda + m; z \\ \lambda \end{matrix}\right] = (1 - z)^{-m-c}\, {}_2F_1\left[\begin{matrix} \lambda - c, -m; z \\ \lambda \end{matrix}\right]$$

multiplying both the sides by $(\lambda)_m\{e_m\}\, x^m/m!$ (where $\{e_m\}$ is any arbitrary sequence) and summing for m from 0 to ∞, we get the following expansion on some manipulation

$$\sum_{j=0}^{\infty} \frac{(c)_j}{j!} z^j \sum_{m=0}^{\infty} \frac{(\lambda + j)_m}{m!} \{e_m\} x^m = (1 - z)^{-c} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \left(\frac{x}{1 - z}\right)^k$$

$$\times \sum_{j=0}^{\infty} \frac{(\lambda - c)_j (\lambda + k)_j}{j! (\lambda)_j} \{e_{k+j}\} \left(\frac{-xz}{1 - z}\right)^j$$

which for $\{e_m\} = [(a_p)]_m/[(b_q)]_m$ yields an expansion due to Srivastava [1970, eqn. (1.5), p. 76] for $c = 1$ and on replacing z by z/c and letting $c \to \infty$ yields another result of Srivastava [1970, eqn. (1.4), p. 75].

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