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SOME RESULTS ON SPACES OF TYPE $LH_{a,\mu}$

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In this paper we give different ways of defining spaces of type $LH_{a,\mu}$. Some operations in spaces of type $LH_{a,\mu}$ and their duals are also discussed.

1. Introduction

In a recent paper the author (Tiwari 1979) has defined spaces of type $LH_{a,\mu}$. In this paper we define some new spaces of type $LH_{a,\mu}$. Some important properties of the spaces of type $LH_{a,\mu}$ are discussed.

Definition 1.1 : *The space* $LH_{a,\mu,\alpha}$.

$\phi \in LH_{a,\mu,\alpha}$; $\alpha \geqslant 0$ if and only if ϕ is a smooth function on $0 < t < \infty$, $0 < x < \infty$ and

$$r^{\mu}_{a,k,q,l}(\phi) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} | e^{at} x^k D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} |$$

$$\leqslant C_q A^k K^{k\alpha};\ k, q, l = 0, 1, \ldots . \qquad \ldots(1.1)$$

Here the constants A and C_q depend on the testing function ϕ. For $k = 0$, $k^{k\alpha} = 1$.

Definition 1.2 : *The space* $L_\beta H_{a,\mu}$

$\phi \in L_\beta H_{a,\mu}$; $\beta \geqslant 0$ if and only if ϕ is a smooth function on $0 < t < \infty$, $0 < x < \infty$ and

$$r^{\mu}_{a,k,q,l}(\phi) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} | e^{at} x^k D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} |$$

$$\leqslant C_k B^l l^{l\beta}, \qquad \ldots(1.2)$$

$$k, q, l = 0, 1, 2 \ldots .$$

As before we set $l^{l\beta} = 1$ for $l = 0$.

For definitions of the spaces $LH^{\beta}_{a,\mu}$ and $LH^{\beta}_{a,\mu,\alpha}$ see Tiwari (1979, pp. 1534–35).

2. Different Ways of Defining Spaces of Type $LH_{a,\mu}$

Definition 2.1 : *The space* $LB_{a,\mu,A}$

$LB_{a,\mu,A}$ is the space of smooth functions $\phi(t, x)$ such that $\phi(t, x) = 0$ for $x > A$ and

$$r^{\mu}_{a,q,l}(\phi) \triangleq \sup_{x,t} \left| e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} \right| < \infty, \quad q, l = 0, 1, \ldots \qquad \ldots(2.1)$$

Definition 2.2 : *The space* $LH_{a,\mu,\alpha,A}$

$LH_{a,\mu,\alpha,A}$ is the space of testing functions ϕ in $LH_{a,\mu}$, such that

$$r^{\mu}_{a,k,q,l} \triangleq \sup_{x,t} \left| e^{at} x^k D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} \right| \leqslant C_{q\delta}(A+\delta)^k k^{k\alpha} \qquad \ldots(2.2)$$

for any $\delta > 0$.

Theorem 2.1 — For $\alpha = 0$, $LH_{a,\mu,\alpha,A} = LB_{a,\mu,A}$

PROOF : Let $\phi \in LH_{a,\mu,\alpha,A}$.

Since $LH_{a,\mu,\alpha,A} \subset LH_{a,\mu,\alpha}$ we have

$$\left| x^k e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} \right| \leqslant C_q A^k K^{k\alpha}. \qquad \ldots(2.3)$$

Dividing both sides by x^k and taking lower bound for k on right-hand side we get

$$\left| e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} \right| \leqslant C_q \inf_k \left[\frac{A^k k^{k\alpha}}{x^k} \right] \qquad \ldots(2.4)$$

$$\leqslant C_q \inf_k \left(\frac{A^k}{x^k} \right), \quad \text{because } \alpha = 0.$$

Now $$\inf_k \left(\frac{A^k}{x^k} \right) = \begin{cases} 1 & \text{if } x \leqslant A \\ 0 & \text{if } x > A. \end{cases}$$

Thus

$$\phi(t, x) = 0 \quad \text{for} \quad x > A. \qquad \ldots(2.5)$$

For $x \leqslant A$ and $\alpha = 0$ we have from (2.2)

$$\left| e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} \right| < \infty. \qquad \ldots(2.6)$$

From (2.5) and (2.6) we have

$$LH_{a,\mu,\alpha,A} \subset LB_{a,\mu,A}.$$

Conversely, let

$$\phi \in LB_{a,\mu,A}$$

so that

$$| x^k e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} | = 0 \text{ for } x > A. \qquad \ldots(2.7)$$

For $x < A$, let

$$\sup_{\substack{0<x\leqslant A \\ 0<t<\infty}} | e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} | = C_q$$

so that

$$\begin{aligned} & | x^k e^{at} D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} | \\ & \leqslant C_q A^k . \frac{(A+\delta)^k}{(A+\delta)^k} \\ & \leqslant C_{q\delta}(A+\delta)^k. \end{aligned} \qquad \ldots(2.8)$$

From (2.7) and (2.8) we have

$$LB_{a,\mu,A} \subset LH_{a,\mu,\alpha,A}$$

This completes the proof of Theorem 2.1.

Theorem 2.2 — Let $\phi \in LH_{a,\mu,\alpha}$, where $\alpha > 0$. Then

$$| D_t^l D_x^q \{x^{-(2\mu+1)/2} \phi\} | \leqslant C_q' \exp(-a'x^{1/\alpha} - at)$$

where a' is some constant depending upon α.

PROOF : The proof of the theorem is similar to the proof of result (6) proved in Gel'fand and Shilov (1968, pp. 170–71).

By definition, the space $LH^{\beta}_{a,\mu,\alpha}$ (see Tiwari 1979, p. 1535) consists of the smooth functions ϕ on $0 < x < \infty$, $0 < t < \infty$ satisfying the inequality

$$\begin{aligned} r^{\mu}_{a,k,q,l}(\phi) & \triangleq \sup_{x,t} | e^{at} x^k D_t^l (x^{-1}D_x)^q \{x^{-(2\mu+1)/2} \phi\} | \\ & \leqslant C A^k B^q K^{k\alpha} q^{q\beta}, \\ & \quad k, q = 0, 1, \ldots . \end{aligned} \qquad \ldots(2.9)$$

From Theorem 2.2 we obtain

$$| D_t^l D_x^q (x^{-(2\mu+1)/2} \phi) | \leqslant C_q B^q q^{q\beta} \exp\{-(a'-\delta) x^{1/\alpha} - at\}, \ \delta > 0.$$

3. Different Ways of Defining Spaces of Type $L^vH_{a,\mu}$

Definition 3.1 : *The space* $L^vB_{a,\mu,\alpha}$

$L^vB_{a,\mu,\alpha}$ is the space of smooth functions $\phi(t, x)$ defined on $-\infty < t < 0$, $0 < x < \infty$ such that $\phi(t, x) = 0$ for $x > A$,

$$i^{\mu}_{a,q,l}(\phi) \triangleq \sup_{\substack{-\infty<t<0\\ 0<x<\infty}} | e^{-at}D^l_t (x^{-1}D_x)^q \{x^{-(2\mu+1)/2}\phi\} | < \infty, \quad q, l = 0, 1, \ldots \qquad \ldots(3.1)$$

and $\phi^v(t, x) = \phi(-t, x) \in LB_{a,\mu,\alpha}$.

Definition 3.2 : *The space* $L^vH_{a,\mu,\alpha,A}$

A smooth function $\phi(t, x)$ defined on $-\infty < t < 0$, $0 < x < \infty$ is in $L^vH_{a,\mu,\alpha,A}$ if $\phi^v(t, x) = \phi(-t, x)$ is in $LH_{a,\mu,\alpha,A}$ and

$$i^{\mu}_{a,k,q,l} \triangleq \sup_{x,t} | e^{-at}x^kD^l_t (x^{-1}D_x)^q \{x^{-(2\mu+1)/2}\phi\} | \leqslant C_{q\delta}(A + \delta)^k k^{k\alpha} \quad \text{for any } \delta > 0. \qquad \ldots(3.2)$$

The following theorems can be easily proved:

Theorem 3.1 — For $\alpha = 0$, $L^vH_{a,\mu,\alpha,A} = L^vB_{a,\mu,A}$.

Theorem 3.2 — If $\phi \in L^vH_{a,\mu,\alpha}$ where $\alpha > 0$.

Then

$$| D^l_t D^q_x \{x^{-(2\mu+1)/2}\phi\} | < C'_q \exp(-a'x^{1/\alpha} + at)$$

where a' is some constant depending upon α.

4. Some Operations in Spaces of Type $LH_{a,\mu}$

(*a*) *Dilatation Operation*

The dilatation operation

$$\phi(t, x) \to \phi(wt, \lambda x); \quad \lambda, w > 0$$

is defined in the space $LH_{a,\mu,\alpha,A}$ and transforms this space into $LH_{aw,\mu,\alpha,(A/\lambda)}$.

Indeed, if $\phi(t, x) \in LH_{a,\mu,\alpha,A}$ so that

$$| e^{at}x^kD^l_t (X^{-1}D_x)^q [x^{-(2\mu+1)/2}\phi] | \leqslant C_{q\delta}(A + \delta)^k k^{k\alpha}. \qquad \ldots(4.1)$$

Then for $\psi(t, x) = \phi(wt, \lambda x)$, we have

$$| e^{(aw)t} x^k D_t^l (x^{-l} D_x)^q \{x^{-(2\mu+1)/2} \psi(t, x)\} |$$

$$= | e^{(aw)t} x^k D_t^l (x^{-l} D_x)^q \{x^{-(2\mu+1)/2} \phi(wt, \lambda x)\} | . \quad ...(4.2)$$

Substituting $wt = T$, $\lambda x = X$ in the right-hand side of (4.2) we have

$$| e^{(aw)t} x^k D_t^l (x^{-l} D_x)^q \{x^{-(2\mu+1)/2} \psi(t, x)\} |$$

$$= \left| e^{aT} \frac{w^l \lambda^{2q+((2\mu+1)/2)}}{\lambda^k} X^k D_T^l (X^{-l} D_X)^q \{X^{-(2\mu+1)/2} \phi(T, X)\} \right|$$

$$\leqslant C_q' \left(\frac{A+\delta}{\lambda}\right)^k k^{k\alpha}. \quad ...(4.3)$$

From (4.3) we easily see that

$\psi(t, x) \in LH_{aw,\mu,\alpha,A/\lambda}$. This shows that the operator $\phi(t, x) \to \phi(wt, x)$ transforms $LH_{a,\mu,\alpha,A}$ into $LH_{aw,\mu,\alpha,A/\lambda}$.

The following results can be easily obtained.

Result 1 — The operation $\phi(t, x) \to \phi(wt, \lambda x)$ is defined in the space $LH_{a,\mu}^{\beta,B}$ and transforms this space into $LH_{aw,\mu}^{\beta,\lambda B}$.

Here $LH_{a,\mu}^{\beta,B}$ is the space of testing functions ϕ in $LH_{a,\mu}^{\beta}$ such that

$$r_{a,k,q,l}^{\mu}(\phi) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} | e^{at} x^k D_t^l (x^{-1} D_x)^q \{x^{-(2\mu+1)/2} \phi\} |$$

$$\leqslant C_{k\rho}(B + \rho)^q q^{q\beta}, \; k, q = 0, 1, 2, ...; \; \rho > 0.$$

Result 2 — The operation $\phi(t, x) \to \phi(wt, \lambda x)$ is defined in the space $LH_{a,\mu,\alpha,A}^{\beta,B}$ and transform this space into $LH_{aw,\mu,\alpha,(A/\lambda)}^{\beta,\lambda B}$, where the space $LH_{a,\mu,\alpha,A}^{\beta,B}$ is the space of testing functions ϕ in $LH_{a,\mu,\alpha}^{\beta}$ such that

$$\sup_{\substack{0<t<\infty \\ 0<x<\infty}} | e^{at} x^k D_t^l (x^{-l} D_x)^q \{x^{-(2\mu+1)/2} \phi\} |$$

$$\leqslant C_{\delta\rho}(A + \delta)^k (B + \rho)^q k^{k\alpha} q^{q\beta}.$$

(*b*) *Differential Operator*

The operation

$$\phi(t, x) \to D_t \phi(t, x)$$

is defined in the space $LH_{a,\mu,\alpha,A}$ and transforms this space into itself.

Indeed if $\phi(t, x) \in LH_{a,\mu,\alpha,A}$ so that

$$| e^{at}x^k D_t^l (x^{-1}D_x)^q [x^{-(2\mu+1)/2} \phi] | \leqslant C_{q,\delta}(A + \delta)^k k^{k\alpha}. \quad \ldots(4.4)$$

Then for $\psi(t, x) = D_t\phi(t, x)$, we have

$$| e^{at}x^k D_t^l (x^{-1}D_x)^q [x^{-(2\mu+1)/2} \psi] |$$

$$= | e^{at}x^k D_t^{l+1} (x^{-1}D_x)^q [x^{-(2\mu+1)/2} \phi] |$$

i.e. $\quad r^{\mu}_{a,k,q,l} (D_t\phi) = r^{\mu}_{a,k,q,l+1} (\phi).$

This shows that the operator $\phi(t, x) \to D_t\phi(t, x)$ transforms the space $LH_{a,\mu,\alpha,A}$ into itself. Similar results can also be proved for the spaces $LH^{\beta,B}_{a,\mu}$ and $LH^{\beta,B}_{a,\mu,\alpha,A}$.

(*c*) *The Operation* $\phi(t, x) \to \phi(-t, x)$

The operation

$$\phi(t, x) \to \phi(-t, x)$$

is an isomorphism from the space $L^v H_{a,\mu,\alpha,A}$ onto the space $LH_{a,\mu,\alpha,A}$.

Indeed,

$$e^{at}x^k D_t^l (x^{-1}D_x)^q [x^{-(2\mu+1)/2} \phi(-t, x)]$$

$$= e^{-aT}x^k(-1)^l D_t^l (x^{-1}D_x)^q [x^{-(2\mu+1)/2} \phi(T, x)]$$

so that $\quad r^{\mu}_{a,k,q,l} \phi(-t, x) \leqslant i^{\mu}_{a,k,q,l} \phi(T, x)$

and therefore $\phi(-t, x) \in LH_{a,\mu,\alpha,A}$, whenever

$$\phi(T, x) \in L^v H_{a,\mu,\alpha,A}.$$

Now the proof is easy.

5. Some Operations in the Dual Spaces

(*a*) *The mapping* $f \to (-D_t f)$

The countable union space $LH(w, \mu, \alpha)$ is defined as (see Tiwari 1979, p. 1534)

$$LH(w, \mu, \alpha) = \bigcup_{v=1}^{\infty} LH_{a_v,\mu,\alpha}.$$

It can be easily seen that the mapping $\phi \to (-D_t\phi)$ is continuous linear mapping from the space $LH(w, \mu, \alpha)$ into itself.

We define the operator D_t on $LH'(w, \mu, \alpha)$ by

$$\langle D_t f, \phi \rangle = \langle f, -D_t\phi \rangle. \quad \ldots(5.1)$$

Here $LH'(w, \mu, \alpha)$ is the dual of the space $LH(w, \mu, \alpha)$. The mapping $f \to D_t(f)$ is the adjoint mapping of the mapping $\phi \to (-D_t\phi)$. Therefore the mapping $f \to (-D_t f)$ is a continuous linear mapping from $LH'(w, \mu, \alpha)$ into itself.

(*b*) *The mapping* $f(t, x) \to f(-t, x)$

For $f \in LH'_{a,\mu,\alpha,A}$ we define the mapping $f(t, x) \to f(-t, x)$ by

$$\langle f(-t, x), \phi(t, x) \rangle \triangleq \langle f(t, x), \phi(-t, x) \rangle \qquad ...(5.2)$$

where $LH'_{a,\mu,\alpha,A}$ is the dual space of the space $LH_{a,\mu,\alpha,A}$. The mapping

$$f(t, x) \to f(-t, x)$$

is the adjoint mapping of the mapping $\phi(t, x) \to \phi(-t, x)$. We have proved in section (4) that $\phi(t, x) \to \phi(-t, x)$ is an isomorphism from $L^v H_{a,\mu,\alpha,A}$ onto the space $LH_{a,\mu,\alpha,A}$. Hence by Theorem (1.10-2) of Zemanian (1968, p. 29) the mapping $f(t, x) \to f(-t, x)$ is an isomorphism from $LH'_{a,\mu,\alpha,A}$ onto the space $L^v H'_{a,\mu,\alpha,A}$.

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