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BIVARIATE DISTRIBUTIONS AND THE MULTIVARIABLE H-FUNCTION

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In this paper, we introduce the probability density function in terms of the multivariable H-function, defined by Srivastava and Panda (1976a, b), and study certain properties of statistical distribution associated with these functions. The generalized phase and radial density functions are obtained by making use of weighted mixture representation, where the independent x- and y-component distributions are not the same. The results established by McNolty and Tomsky (1972), Saxena and Sethi (1973), etc., follow as special cases of our findings.

1. INTRODUCTION

During the study of a military research problem, McNolty and Tomsky (1972) considered phase and radial densities. Indeed, phase distribution arises in the problem of reconstructing a phase density function from the given radial (or amplitude) density and also in describing bivariate populations directly in terms of phase (azimuth) and radial distributions rather than their cartesian component distributions. During the analysis of the distributions associated with the random variables x, y, r and ϕ, it is desirable to include the cases in which the distributions of x and y components are not necessarily the same. Consequently, distributions of the components are restricted to that of even functions of the variates. Saxena and Sethi (1973) and Srivastava (1976) have recently generalized these problems with the help of distributions associated with generalized hypergeometric functions and (Fox's) H-function respectively.

In this paper, we consider a very general case of McNolty and Tomsky, when x and y components have some probability laws defined in terms of the multivariable H-function introduced and studied by Srivastava and Panda [1976a, p. 271, eqn. (4.1) *et seq.*]. The probability density function considered here contains (as particular cases) a large variety of such functions introduced in the literature from time to time. Thus the results established here will cover a wide range of probability laws for x and y components. The results of aforementioned authors will be the special cases of our findings.

The multivariable H-function occurring in this paper is a special case of the general H-function of several complex variables introduced and studied by Srivastava and Panda (1976a, 1976b). The parameters of this function will be displayed here in the following contracted notation, which slightly differs from that of Srivastava and Panda [1976a, p. 130, eqn. (1.3)] :

$$H^{(1)}[x_1, \ldots, x_k] = H^{0,n:1,n_1;\cdots;1,n_k}_{p,q:p_1,q_1;\cdots;p_k,q_k}\left[\begin{matrix} x_1 \\ \vdots \\ x_k \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j', \ldots, \alpha_j^{(k)})_{1,p} : \\ (b_j;\beta_j', \ldots, \beta_j^{(k)})_{1,q} : \end{matrix} \begin{matrix} (c_j', \epsilon_j')_{1,p_1} \quad ; \ldots; \quad (c_j^{(k)}, \epsilon_j^{(k)})_{1,p_k} \\ (0,1), (d_j', \delta_j')_{2,q_1}; \ldots; (0,1), (d_j^{(k)}, \delta_j^{(k)})_{2,q_k} \end{matrix}\right]$$

$$= (2\pi\omega)^{-k} \int_{L_1} \cdots \int_{L_k} \phi(s_1, \ldots, s_k) \prod_{i=1}^{k} \{\theta_i(s_i)\, \Gamma(-s_i)\, (x_i)^{s_i}\, ds_i\} \qquad \ldots(1.1)$$

where $\omega = \sqrt{-1}$, and

$$\phi(s_1, \ldots, s_k) = \prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{k} \alpha_j^{(i)} s_i) \times \left[\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{k} \alpha_j^{(i)} s_i) \prod_{j=1}^{q} \Gamma\left(1 - b_j + \sum_{i=1}^{k} \beta_j^{(i)} s_i\right)\right]^{-1} \qquad \ldots(1.2)$$

$$\theta_i(s_i) = \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \epsilon_j^{(i)} s_i) \times \left[\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \epsilon_j^{(i)} s_i) \prod_{j=2}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i)\right]^{-1}$$

$$(i = 1, \ldots, k) \qquad \ldots(1.3)$$

(i) in the superscript stands for the number of primes, e.g., $b^{(1)} = b'$, $b^{(2)} = b''$, and so on; $(a_j;\alpha_j', \ldots, \alpha_j^{(k)})_{1,p}$ would abbreviate $(a_1;\alpha_1', \ldots, \alpha_1^{(k)}), \ldots, (a_p;\alpha_p', \ldots, \alpha_p^{(k)})$ and $(c_j, \epsilon_j)_{n+1,p}$ the $(p-n)$ parameter sequence $(c_{n+1}, \epsilon_{n+1}), \ldots, (c_p, \epsilon_p)$ for integers n and p such that $0 \leqslant n \leqslant p$, and so on.

The conditions of convergence for the multiple integral (1.1), and other details for the multivariable H-function can be found in the papers by Srivastava and Panda (1976a, 1976b).

The following result will be required in the course of our analysis :

$$H^{(1)}[x_1, \ldots, x_k] = \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_k=0}^{\infty} \phi(\nu_1, \ldots, \nu_k) \prod_{i=1}^{k} \left\{\frac{\theta_i(\nu_i)\,(-x_i)^{\nu_i}}{\nu_i!}\right\} \qquad \ldots(1.4)$$

where $\phi(v_1, \ldots, v_k)$ and $\theta_i(v_i)$ are defined by (1.2) and (1.3) respectively. The above result follows easily from a series expansion given by Saxena [1977, p. 225, eqn. (4.1)].

2. A General Probability Model

Theorem 1 — Let $a > 0$, $\lambda > 0$, $\sigma_i > 0$, $U_i > 0$, where

$$U_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \epsilon_j^{(i)} + \sum_{j=n_i+1}^{p_i} \epsilon_j^{(i)} - \sum_{j=2}^{q_i} \delta_j^{(i)} + 1, \quad (i = 1, \ldots, k). \qquad \ldots(2.1)$$

Then

$$f(x)\, dx = S a^{\lambda} \mid x \mid^{2\lambda-1} e^{-ax^2} H^{(1)} [z_1 x^{2\sigma_1}, \ldots, z_k x^{2\sigma_k}]\, dx, \quad (-\infty < x < \infty) \qquad \ldots(2.2)$$

defines a general probability model, provided

$$S^{-1} = H^{0,n+1:1,n_1;\cdots;1,n_k}_{p+1,q:p_1,q_1;\cdots;p_k,q_k} \left[\begin{array}{c} z_1 a^{-\sigma_1} \\ \vdots \\ z_k a^{-\sigma_k} \end{array} \middle| \begin{array}{l} (1-\lambda; \sigma_1, \ldots, \sigma_k), (a_j; \alpha_j', \ldots, \alpha_j^{(k)})_{1,p} : (c_j', \epsilon_j')_{1,p_1}; \ldots; (c_j^{(k)}, \epsilon_j^{(k)})_{1,p_k} \\ (b_j; \beta_j', \ldots, \beta_j^{(k)})_{1,q} : (0, 1), (d_j', \delta_j')_{2,q_1}; \ldots; (0, 1), (d_j^{(k)}, \delta_j^{(k)})_{2,q_k} \end{array} \right] \qquad \ldots(2.3)$$

Since there are a number of parameters of the multivariable H-function in (2.2), there exist a set of parameters for which this function is nonnegative. Hence, we shall assume that the multivariable H-function that is used here, is nonnegative.

The proof of the Theorem 1 is quite straightforward and therefore we omit the details.

Now, suppose the probability density function of the x-component is given by (2.2). Similarly, let the probability density function for the y-component is given by

$$f(y)\, dy = T a^{\mu} \mid y \mid^{2\mu-1} e^{-ay^2} H^{(2)} [z_1 y^{2\sigma_1}, \ldots, z_k y^{2\sigma_k}]\, dy \qquad \ldots(2.4)$$

where

$$H^{(2)} [x_1, \ldots, x_k] = H^{0,n:1,n_1;\cdots;1,n_k}_{p,q:p_1,q_1;\cdots;p_k,q_k} \left[\begin{array}{c} x_1 \\ \vdots \\ x_k \end{array} \middle| \begin{array}{l} (a_j'; A_j', \ldots, A_j^{(k)})_{1,p} : (e_j', E_j')_{1,p_1}; \ldots; (e_j^{(k)}, E_j^{(k)})_{1,p_k} \\ (b_j'; B_j', \ldots, B_j^{(k)})_{1,q} : (0, 1), (f_j', F_j')_{2,q_1}; \ldots; (0, 1), (f_j^{(k)}, F_j^{(k)})_{2,q_k} \end{array} \right] \qquad \ldots(2.5)$$

$$T^{-1} = H^{0,n+1:1,n_1;\cdots;1,n_k}_{p+1,q:p_1,q_1;\cdots;p_k,q_k}\left[\begin{matrix} z_1a^{-\sigma_1} \\ \vdots \\ z_ka^{-\sigma_k}\end{matrix}\middle| \begin{matrix}(1-\mu;\sigma_1,\ldots,\sigma_k),(a'_j;A'_j,\ldots,A^{(k)}_j)_{1,p}: \\ (b'_j;B'_j,\ldots,B^{(k)}_j)_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(e'_j,E'_j)_{1,p_1};\ldots;(e^{(k)}_j,E^{(k)}_j)_{1,p_k} \\ (0,1),(f'_j,F'_j)_{2,q_1};\ldots;(0,1\,(f^{(k)}_j,F^{(k)}_j)_{2,q_k}\end{matrix}\right] \qquad \ldots(2.6)$$

and the conditions modified appropriately, given in Theorem 1 are satisfied.

From (2.2) and (2.4), it is clear that the distributions for $u = x^2$ and $v = y^2$ will be given by

$$g_1(u)\,du = \frac{S}{2}\,a^{\lambda}u^{\lambda-1}e^{-au}H^{(1)}\,[z_1u^{\sigma_1},\ldots,z_ku^{\sigma_k}]\,du,\ u>0 \qquad \ldots(2.7)$$

$$g_2(v)\,dv = \frac{T}{2}\,a^{\mu}v^{\mu-1}e^{-av}H^{(2)}\,[z_1v^{\sigma_1},\ldots,z_kv^{\sigma_k}]\,dv,\ v>0. \qquad \ldots(2.8)$$

3. Joint Distributions

Theorem 2 — Let the distributions for u and v are as given by (2.7) and (2.8) respectively. Then the distribution for $z = u/v$ is defined by

$$f_1(z)\,dz = \frac{STz^{\lambda-1}}{4(1+z)^{\lambda+\mu}}\sum_{v_1=0}^{\infty}\cdots\sum_{v_k=0}^{\infty}\phi(v_1,\ldots,v_k)\prod_{i=1}^{k}\left\{\frac{\theta_i(v_i)\,(-z_i)^{v_i}}{v_i!}\right.$$

$$\left.\times\left(\frac{z}{a(1+z)}\right)^{\sigma_iv_i}\right\}F_{v_1,\ldots,v_k}\,[z_1\{a(1+z)\}^{-\sigma_1},\ldots,$$

$$\times\ z_k\{a(1+z)\}^{-\sigma_k}]\,dz,\ (z>0) \qquad \ldots(3.1)$$

where

$$F_{v_1,\ldots,v_k}\,[x_1,\ldots,x_k] = H^{0,n+1:1,n_1;\cdots;1,n_k}_{p+1,q:p_1,q_1;\cdots;p_k,q_k}\left[\begin{matrix} x_1 \\ \vdots \\ x_k\end{matrix}\middle| \begin{matrix}(1-\lambda-\mu-\sum_{i=1}^{k}\sigma_iv_i;\sigma_1,\ldots,\sigma_k),(a'_j;A'_j,\ldots,A^{(k)}_j)_{1,p}: \\ (b'_j;B'_j,\ldots,B^{(k)}_j)_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(e'_j,E'_j)_{1,p_1};\ldots;(e^{(k)}_j,E^{(k)}_j)_{1,p_k} \\ (0,1),(f'_j,F'_j)_{2,q_1};\ldots;(0,1),(f^{(k)}_j,F^{(k)}_j)_{2,q_k}\end{matrix}\right] \qquad \ldots(3.2)$$

$\phi(v_1, \ldots, v_k)$ and $\theta_i(v_i)$ are defined by (1.2) and (1.3) respetively. The joint distribution (3.1) is valid under the following conditions:

(i) $\sigma_i > 0$, $\min(U_i, T_i) > 0$, where

$$T_i = -\sum_{j=n+1}^{p} A_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} + \sum_{j=1}^{n_i} E_j^{(i)} - \sum_{j=n_i+1}^{p_i} E_j^{(i)} - \sum_{j=2}^{q_i} F_j^{(i)} + 1, \quad (i = 1, \ldots, k) \qquad \ldots(3.3)$$

and U_i is given by (2.1).

(ii) $\lambda + \mu > 0$ and $a > 0$.

(iii) The multiple series on the right-hand side of (3.1) converges absolutely.

Also $f_1(z) = 0$, elsewhere.

PROOF : If we set $z = u/v$ and $t = v$, then the joint distribution of z and t is given by

$$f_1(z, t) = g(u, v) \left| \frac{\partial(u, v)}{\partial(z, t)} \right| = g_1(u)\, g_2(v)\, v. \qquad \ldots(3.4)$$

Equation (3.4) in conjunction with (2.7) and (2.8) yields

$$f_1(z) = \int_0^\infty f(z, t)\, dt = \frac{ST}{4} a^{\lambda+\mu} z^{\lambda-1} \int_0^\infty t^{\lambda+\mu-1} e^{-a(1+z)t}$$
$$\times H^{(1)}[z_1(zt)^{\sigma_1}, \ldots, z_k(zt)^{\sigma_k}]\, H^{(2)}[z_1 t^{\sigma_1}, \ldots, z_k t^{\sigma_k}]\, dt. \qquad \ldots(3.5)$$

Now, using the series expansion given by (1.4) for $H^{(1)}$-function in (3.5), changing the order of integration and summation (which is justified under the conditions stated with the theorem), and evaluating the t-integral so obtained with the help of a known result [Panda, 1977, p. 120, eqn. (3.14)], we arrive at the right-hand side of (3.1).

Next, we mention below some useful and interesting deductions of Theorem 2 :

Corollary 1 — If we set $z = \frac{s}{1-s}$, then the distribution of s can be obtained from (3.1), which is given below

$$f_2(s)\, ds = \frac{ST}{4} s^{\lambda-1} (1-s)^{\mu-1} \sum_{v_1=0}^{\infty} \cdots \sum_{v_k=0}^{\infty} \phi(v_1, \ldots, v_k) \prod_{i=1}^{k} \left[\frac{\theta_i(v_i)}{v_i!} \right.$$
$$\times \left. \left\{ -\left(\frac{s}{a}\right)^{\sigma_i} z_i \right\}^{v_i} \right] F_{v_1,\ldots,v_k}\left[z_1 \left(\frac{1-s}{a}\right)^{\sigma_1}, \ldots, z_k \left(\frac{1-s}{a}\right)^{\sigma_k} \right] ds \ldots(3.6)$$

for $0 \leqslant s \leqslant 1$, and $f_2(s) = 0$, elsewhere. The s-distribution defined by (3.6) is valid, if the set of conditions (i) and (ii) given with Theorem 2 are satisfied and the series on the right-hand side of (3.6) is absolutely convergent.

Corollary 2 — Again, if we take $s = \cos^2 \phi$ in (3.6), we get the random phase (ϕ) distribution:

$$f_3(\phi)\, d\phi = \frac{ST}{2} \mid \cos \phi \mid^{2\lambda-1} \mid \sin \phi \mid^{2\mu-1} \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_k=0}^{\infty} \phi(\nu_1, \ldots, \nu_k)$$

$$\times \prod_{i=1}^{k} \left\{ \frac{\theta_i(\nu_i)}{\nu_i!} \left(\frac{-z_i \mid \cos \phi \mid^{2\sigma_i}}{a^{\sigma_i}} \right)^{\nu_i} \right\}$$

$$\times F_{\nu_1,\ldots,\nu_k} \left[z_1 \left(\frac{\sin^2 \phi}{a} \right)^{\sigma_1}, \ldots, z_k \left(\frac{\sin^2 \phi}{a} \right)^{\sigma_k} \right] d\phi \qquad \ldots(3.7)$$

for $0 < \phi < 2\pi$, and $f_3(\phi) = 0$, elsewhere. Incidentally, the conditions of existence for $f_3(\phi)$ are the same as those for $f_2(s)$ given in Corollary 1.

The theorem below provides the 'radial distribution':

Theorem 3 — Let the distributions for u and v are defined by (2.7) and (2.8), respectively. Then the distribution for $r = \sqrt{u + v}$ is given by

$$f_4(r)\, dr = \frac{ST}{2} a^{\lambda+\mu} e^{-ar^2} r^{2\lambda+2\mu-1} \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_k=0}^{\infty} \phi(\nu_1, \ldots, \nu_k)$$

$$\times \Gamma\left(\lambda + \sum_{i=1}^{k} \sigma_i \nu_i\right) \prod_{i=1}^{k} \left\{ \frac{\theta_i(\nu_i)}{\nu_i!} (-z_i r^{2\sigma_i})^{\nu_i} \right\} H^{0,n+1:1,n_1;\cdots;1,n_k}_{p+1,q+1:p_1,q_1;\cdots;p_k,q_k}$$

$$\left[\begin{matrix} z_1 r^{2\sigma_1} \\ \vdots \\ z_k r^{2\sigma_k} \end{matrix} \left| \begin{matrix} (1-\mu; \sigma_1, \ldots, \sigma_k), (a_j'; A_j', \ldots, A_j^{(k)})_{1,p} : \\ (1-\lambda-\mu-\sum_{i=1}^{k} \sigma_i \nu_i; \sigma_1, \ldots, \sigma_k), (b_j'; B_j', \ldots, B_j^{(k)})_{1,q} : \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (e_j', E_j')_{1,p_1} ; \ldots ; (e_j^{(k)}, E_j^{(k)})_{1,p_k} \\ (0, 1), (f_j', F_j')_{2,q_1}; \ldots; (0, 1), (f_j^{(k)}, F_j^{(k)})_{2,q_k} \end{matrix} \right] dr \qquad \ldots(3.8)$$

where $r > 0$, $\lambda > 0$, $\mu > 0$, $a > 0$, the set of conditions (i) given with Theorem 2 holds and the series on the right-hand side of (3.8) converges absolutely. Also $f_4(r) = 0$, elsewhere.

PROOF : Using the transformation $r = (u + v)^{1/2}$ and $t = v$, then $\left|\frac{\partial(u, v)}{\partial(r, t)}\right| = 2r$ and the joint density of r and t will be given by

$$f(r, t) = g(u, v)\left|\frac{\partial(u, v)}{\partial(r, t)}\right| = 2g_1(u)\, g_2(v)\, r \qquad \text{...(3.9)}$$

where $0 < t < r^2$ and $0 \leqslant r < \infty$.

Hence, on using (2.7) and (2.8), we have the distribution for r, that is

$$f_4(r)\, dr = \int_0^{r^2} f(r, t)\, dt = \frac{ST}{2} a^{\lambda+\mu} r e^{-ar^2} \int_0^{r^2} t^{\mu-1}(r^2 - t)^{\lambda-1}$$

$$\times H^{(1)}[z_1(r^2 - t)^{\sigma_1}, \ldots, z_k(r^2 - t)^{\sigma_k}]\, H^{(2)}[z_1 t^{\sigma_1}, \ldots, z_k t^{\sigma_k}]\, dt. \qquad \text{...(3.10)}$$

Putting $t = r^2 x$, writing the series expansion (1.4) for $H^{(1)}$-function in (3.10), changing the order of integration and summation and then evaluating the x-integral thus obtained with the help of (1.1) and the well-known definition of beta function, we arrive at the desired result (3.8).

4. SPECIAL CASES

At the outset we should remark that the multivariable H-function defined by (1.1) includes a large variety of elementary special functions involving one or more variables as its particular cases. Thus our probability model given by (2.2) is quite general in nature and from it almost all the known statistical distributions can be derived as specialized or limiting cases of our distribution (2.2). For example, if we set $k = 2$, $n = p = q = 0 = n_2 = p_2$, $q_2 = \sigma_1 = 1$ and $z_2 \to 0$ in (2.2), and apply a known result by Goyal [1975, p. 123, eqn. (3.5)] therein, we get the probability model which differs slightly from that discussed by Srivastava [1976, p. 231, eqn. (5.1)] in the following form

$$f(x) = Pa^{\lambda}\, |\, x\, |^{\,2\lambda-1}\, e^{-ax^2} H^{1,n}_{p,q}\left[zx^2 \,\middle|\, \begin{matrix} (c_j, \epsilon_j)_{1,p} \\ (0, 1), (d_j, \delta_j)_{2,q} \end{matrix} \right] \qquad \text{...(4.1)}$$

where $-\infty < x < \infty$, $\lambda > 0$, $a > 0$, and

$$P^{-1} = H^{1,n+1}_{p+1,q}\left[\frac{z}{a} \,\middle|\, \begin{matrix} (1 - \lambda, 1), (c_j, \epsilon_j)_{1,p} \\ (0, 1), (d_j, \delta_j)_{2,q} \end{matrix} \right] \qquad \text{...(4.2)}$$

Further, if we let $n = p$, all ϵ's and δ's equal to one in (4.1), and using a known result (Mathai and Saxena 1978, p. 151), we shall easily arrive at the generalized hypergeometric distribution studied by Saxena and Sethi [1973, p. 172, eqn. (1)], which evidently includes the confluent hypergeometric distribution discussed by McNolty and Tomsky [1972, p. 253, eqn. (35)], as its particular case. Also, the various results established by these authors will be the special cases of our Theorems 2 and 3 and Corollaries 1 and 2.

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