Indian J. pure appl. Math., **12**(3) : 388-392, March 1981

INTEGRALS INVOLVING THE GENERALIZED LAURICELLA FUNCTION

P. C. MUNOT AND P. A. PADMANABHAM

Department of Mathematics, University of Jodhpur, Jodhpur 342001

(*Received* 26 *December* 1979; *after revision* 7 *July* 1980)

In the present paper the authors have evaluated two definite integrals involving the generalized Lauricella function of several variables defined recently by Srivastava and Daoust (1969), of which one is a finite and the other is an infinite integral. The general character of these results is obvious by the fact that, by restricting the parameters and the number of variables in the generalized Lauricella function, these results can be fairly easily particularized to various finite integrals involving functions like F_A, F_B, F_C, F_D and others.

INTRODUCTION

As usual, the symbol (a) denotes the sequence of A parameters $a_1, a_2, \ldots, a_A$, with a similar interpretation of (b), (c), etc., whereas $\Delta(k, c)$ represents the set of k parameters

$$\frac{c}{k}, \frac{c+1}{k}, \ldots, \frac{c+k-1}{k}, k \geqq 1.$$

Srivastava and Daoust (1969, p. 454), in an attempt to generalize Lauricella's four hypergeometric functions F_A, F_B, F_C and F_D, have recently defined the generalized Lauricella function of several variables by

$$F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi^{(')}],\ldots,[(b^{(r)}):\phi^{(r)}]\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta^{(')}],\ldots,[(d^{(r)}):\delta^{(r)}]\end{matrix}\; z_1,\ldots,z_r\right]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_X \prod_{j=1}^{B'}(b_j')_Y \cdots \prod_{j=1}^{B^{(r)}}(b_j^{(r)})_Z}{\prod_{j=1}^{C}(c_j)_L \prod_{j=1}^{D'}(d_j')_M \cdots \prod_{j=1}^{D^{(r)}}(d_j^{(r)})_N}$$

$$\times \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!}, \qquad \ldots(1.1)$$

X, Y, Z, L, M, N (1.1) should be read as follows :

$$X = \sum_{i=1}^{r} m_i\theta_j^{(i)}, \quad Y = m_1\phi_j', \quad Z = m_r\phi_j^{(r)}$$

$$L = \sum_{i=1}^{r} m_i\psi_j^{(i)}, \quad M = m_1\delta_j', \quad N = m_r\delta_j^{(r)}$$

which converges when

$$\Delta_i = 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D(i)} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B(i)} \phi_j^{(i)} \geqq 0 \qquad \ldots(1.2)$$

the equality holds when $|z_i| < \rho_i$, $i = 1, 2, \ldots, r$, where ρ_i is given by

$$\rho_i = \min_{\mu_1, \mu_2, \ldots > 0} \{E_i\}$$

and E's are given by :

$$E_i = (\mu_i)^{1 + \sum_{j=1}^{D(i)} \delta_j^{(i)} - \sum_{j=1}^{B(i)} \phi_j^{(i)}} \times \frac{\prod_{j=1}^{C} \left(\sum_{i=1}^{r} \mu_i \psi_j^{(i)}\right)^{\psi_j^{(i)}} \prod_{j=1}^{D(i)} (\delta_j^{(i)})^{\delta_j^{(i)}}}{\prod_{j=1}^{A} \left(\sum_{i=1}^{r} \mu_i \theta_j^{(i)}\right)^{\theta_j^{(i)}} \prod_{j=1}^{B(i)} (\phi_j^{(i)})^{\phi_j^{(i)}}}$$

§2. The main result to be established is:

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}[1+cx+d(1-x)]^{-\lambda-\mu}\, {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \mu;\end{matrix}\ \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$\times F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[\begin{matrix}[(a):\theta', \ldots, \theta^{(r)}] : [(b'):\phi']; \ldots; [(b^{(r)}):\phi^{(r)}] \\ [(c):\psi', \ldots, \psi^{(r)}] : [(d'):\delta']; \ldots; [(d^{(r)}):\delta^{(r)}]\end{matrix};\right.$$

$$\left. z_1\left[\frac{x(1+c)}{1+cx+d(1-x)}\right]^n \ldots z_r\left[\frac{x(1+c)}{1+cx+d(1-x)}\right]^n\right] dx$$

$$= L \cdot F^{A+2n:B';\ldots;B^{(r)}}_{C+2n:D';\ldots;D^{(r)}}\left[\begin{matrix}[(a):\theta', \ldots, \theta^{(r)}]; [\Delta(n, \lambda) : 1, 1, \ldots, 1]; \\ [c):\psi', \ldots, \psi^{(r)}]; [\Delta(n, \lambda+\mu-\alpha) : 1, 1, \ldots, 1];\end{matrix}\right.$$

$$\begin{matrix}[\Delta(n, \lambda+\mu-\alpha-\beta) : 1, 1, \ldots, 1] : [(b'):\phi']; \ldots; \\ [\Delta(n, \lambda+\mu-\beta) \qquad : 1, 1, \ldots, 1] : [d'):\delta']; \ldots;\end{matrix}$$

$$\left.\begin{matrix}[(b^{(r)}):\phi^{(r)}] \\ [(d^{(r)}):\delta^{(r)}]\end{matrix}; z_1, z_2, \ldots, z_r\right] \qquad \ldots(2.1)$$

where

$$L = \{\Gamma(\lambda)\,\Gamma(\lambda+\mu-\alpha-\beta)\}/\{\Gamma(\lambda+\mu-\alpha)\,\Gamma(\lambda+\mu-\beta) \times (1+c)^{\lambda}(1+d)^{\mu}\}. \qquad \ldots(2.2)$$

valid for $\operatorname{Re}(\mu) > 0$, $\operatorname{Re}(\lambda) > 0$, $\operatorname{Re}(\mu - \alpha - \beta) > 0$ and the conditions of existence of the Lauricella functions involved.

PROOF : On expressing the Lauricella function on left hand side of (2.1) in its equivalent series form and then changing the order of integration and summation, which is permissible under the conditions stated with it, it can be easily seen that this side becomes

$$\sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_X \prod_{j=1}^{B'} (b'_j)_Y \cdots \prod_{j=1}^{B^{(r)}} (b_j^{(r)})_Z}{\prod_{j=1}^{C} (c_j)_L \prod_{j=1}^{D'} (d'_j)_M \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_N}$$

$$\times \frac{[z_1(1+c)^n]^{m_1}}{m_1!} \cdots \frac{[z_r(1+c)^n]^{m_r}}{m_r!} \int_0^1 x^{\lambda+n(m_1+\cdots+m_r)-1} (1-x)^{\mu-1}$$

$$\times [1+cx+d(1-x)]^{-\lambda-\mu} {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \mu\end{matrix}; \frac{(1-x)(1+d)}{[1+cx+d(1-x)]}\right] dx.$$

The values of subscripts X, Y, Z, L, M, N are as in eqn. (1.1).

Now evaluating the inner integral with the help of MacRobert's result (1962) viz.,

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}[1+cx+d(1-x)]^{-\lambda-\mu}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \mu\end{matrix}; \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right] dx$$

$$= (1+c)^{-\lambda}(1+d)^{-\mu} \frac{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu-\alpha-\beta)}{\Gamma(\lambda+\mu-\alpha)\Gamma(\lambda+\mu-\beta)} \quad \ldots(2.3)$$

valid for $\operatorname{Re}(\mu) > 0$, $\operatorname{Re}(\lambda) > 0$, $\operatorname{Re}(\mu - \alpha - \beta) > 0$, and the result (Rainville 1960) viz.,

$$(\alpha)_{kt} = k^{kt} \prod_{i=1}^{k} \left(\frac{\alpha+i-1}{k}\right)_t \quad \ldots(2.4)$$

in succession we immediately get the right hand side of (2.1). Similarly, on using the result (MacRobert 1961),

$$\int_0^\infty e^{-px}x^{\lambda-1}E(\rho, \mu : px)\, dx = \frac{\Gamma(\rho)\Gamma(\mu)\Gamma(\lambda+\rho)\Gamma(\lambda+\mu)}{p^{\lambda}\Gamma(\lambda+\rho+\mu)} \quad \ldots(2.5)$$

valid for $\operatorname{Re}(\lambda+\rho) > 0$, $\operatorname{Re}(\lambda+\mu) > 0$, $\operatorname{Re}(p) > 0$, we can also obtain the result

$$\int_0^\infty e^{-px} x^{\lambda-1} E(\rho, \mu : px)$$

$$\times F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],\ldots,[(b^{(r)}):\phi^{(r)}]\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'],\ldots,[(d^{(r)}):\delta^{(r)}]\end{matrix};z_1x^n,\ldots,z_rx^n\right]dx$$

$$= K \,.\, F^{A+2n:B';\cdots;B^{(r)}}_{C+\ n:D';\cdots;D^{(r)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}];[\Delta(n,\lambda+\rho):1,\ldots,1];\\ [(c):\psi',\ldots,\psi^{(r)}];[\Delta(n,\lambda+\rho+\mu):1,1,\ldots,1];\end{matrix}\right.$$

$$\begin{matrix}[\Delta(n,\lambda+\mu):1,1,\ldots,1]:[(b'):\phi'];\ldots[(b^{(r)}):\phi^{(r)}];\\ \text{—}\qquad\qquad :[(d'):\delta'];\ldots[(d^{(r)}):\delta^{(r)}];\end{matrix}$$

$$\left. z_1\left(\frac{n}{p}\right)^n,\ldots,z_r\left(\frac{n}{p}\right)^n\right] \qquad \ldots(2.6)$$

where

$$K = \{\Gamma(\rho)\,\Gamma(\mu)\,\Gamma(\lambda+\rho)\,\Gamma(\lambda+\mu)\}/\{p^\lambda\Gamma(\lambda+\rho+\mu)\} \qquad \ldots(2.7)$$

valid for Re $(\lambda+\rho) > 0$, Re $(\lambda+\mu) > 0$, Re $(p) > 0$ and the conditions under which the generalized Lauricella functions exist. The result (2.6) can also be obtained as a special form of a known result due to Panda [1977, p. 122, eqn. (4.8)]. {See also Panda (1973) for several other integrals involving the (Srivastava-Daoust) generalized Lauricella functions.}

§3. *Some special cases of* (2.1) — (i) On setting each θ's, ψ's, ϕ's and δ's equal to unity and $r = 2$, the results (2.1) and (2.6), will give us the results involving Kampé de Fériet function of two variables.

(ii) Again on taking each θ's, ψ's, ϕ's and δ's equal to unity and $A = C = 0$ along with $r = 2$, the result (2.1), yields to,

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}[1+cx+d(1-x)]^{-\lambda-\mu}\,{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \mu\end{matrix};\frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$\times\ {}_{B'}F_{D'}\left[(b');(d');z_1\left(\frac{x(1+c)}{1+cx+d(1-x)}\right)^n\right]$$

$$\times\ {}_{B''}F_{D''}\left[(b'');(d'');z_2\left(\frac{x(1+c)}{1+cx+d(1-x)}\right)^n\right]dx$$

$$= L \,.\, F^{2n:B';B''}_{2n:D';D''}\left[\begin{matrix}\Delta(n,\lambda);\Delta(n,\lambda+\mu-\alpha-\beta)\quad :(b');(b'')\\ \Delta(n,\lambda+u-\alpha),\Delta(n,\lambda+\mu-\beta):(d');(d'')\end{matrix};z_1,z_2\right] \qquad \ldots(3.1)$$

valid for Re $(\mu) > 0$, Re $(\lambda) > 0$, Re $(\mu-\alpha-\beta) > 0$ and the conditions under which right hand converges. Here the function on the right is the Kampé de Fériet function of two variables (Appell and Kampé de Fériet 1926) in the contracted notation of Burchnall and Chaundy (1941) and the function on the left is generalized hypergeometric function (Rainville 1960).

(iii) On setting each of the θ's, ψ's, ϕ's and δ's equal to unity, $r = 2$ and $B = D = 0$, in (2.1), we get

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}[1+cx+d(1-x)]^{-\lambda-\mu}\, {}_2F_1\left[\begin{matrix}\alpha, \beta;\\ \mu\end{matrix}\; \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$\times\, {}_AF_C\left[(a); (c); (z_1+z_2)\left(\frac{x(1+c)}{1+cx+d(1-x)}\right)^n\right] dx$$

$$= L.\; {}_{A+2n}F_{C+2n}\left[\begin{matrix}(a), \Delta(n,\lambda), \Delta(n, \lambda+\mu-\alpha-\beta)\\ (c), \Delta(n, \lambda+\mu-\alpha), \Delta(n, \lambda+\mu-\beta)\end{matrix}; (z_1+z_2)\right] \quad \ldots(3.2)$$

valid for $\mathrm{Re}\,(\mu) > 0$, $\mathrm{Re}\,(\lambda) > 0$, $\mathrm{Re}\,(\mu - \alpha - \beta) > 0$, and the conditions under which right hand converge.

As per remark given in introduction, we can easily evaluate various integrals, particularly those involving F_A, F_B, F_C, F_D and many others hypergeometric functions of one and more variables, as special cases of (2.1) and (2.6). For the sake of brevity, we are not mentioning these results here.

ACKNOWLEDGEMENT

The authors wish to thank Professor R. S. Kushwaha for his constant encouragements, during the preparation of this paper. Thanks are also due to the referee for some useful suggestions.

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