Indian J. pure appl. Math., **12**(3) : 393-399, March 1981

GENERALIZED SOLUTION OF A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM OF ASTROPHYSICS

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(*Received* 9 *May* 1980)

In certain astrophysical problems there arises a partial differential equation of second order with a boundary condition and two further conditions at an interior point. The equation contains two physical parameters, is not in normal form, and the domain is not specified in advance. A particular value of one of the physical parameters is considered (polytropic index is taken to be 2). It is shown that one of the interior conditions can be satisfied by scaling the domain. The other interior condition needs further investigation. The equation is then put in normal form by an elementary transformation. The main result is as follows. For any smooth domain there exist arbitrarily small values of the other physical parameter such that the transformed equation and boundary condition have a solution in the distributional sense. The proof depends heavily on ideas of Levinson (1962).

INTRODUCTION

The following partial differential equation on a plane bounded domain Ω containing the origin occurs in some problems of astrophysics (Sood 1972, Trehan and Billings 1971, Trehan and Uberoi 1972) :

$$\Delta\Theta = -\Theta^N - h\left[\Theta^N + (1-\mu^2)\,N\Theta^{N-1}\left(\xi\frac{\partial\Theta}{\partial\xi} - \mu\frac{\partial\Theta}{\partial\mu}\right) + \tfrac{1}{4}\xi^2(1-\mu^2)\,\Delta\Theta^N\right]. \quad ...(1)$$

Here ξ, θ are polar coordinates, $\mu = \cos\theta$, Δ is the Laplacian and h, N are physical constants. The unknown function Θ is required to satisfy the boundary condition

$$\Theta\text{ (boundary)} = 0 \quad ...(2)$$

as well as the following conditions at the origin 0

$$\Theta(0) = 1,\ \frac{\partial\Theta}{\partial\xi}(0) = 0. \quad ...(3)$$

The first part of (3) rules out the trivial solution $\Theta \equiv 0$.

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We restrict ourselves to the case $N = 2$. It can then be verified by a direct computation that if Θ is a solution of (1) and (2) on Ω, then the function $a^2\Theta(a\xi, \theta)$ on the scaled domain $a\Omega$ is also a solution of (1) and (2). One may therefore regard the first part of (3) as restricting the size of the domain. Since the domain is not specified in advance, but is expected to be determined by the (existence of the) solution, it seems reasonable to expect that the second part of (3) restricts the shape of the domain. In the present work we consider (1) and (2) on a domain with a sufficiently smooth boundary; the question of which domains have a solution satisfying the second part of (3) is not taken up here. We first derive the corresponding equation for a function related to Θ, assuming that the latter is twice differentiable; for the new equation we prove that generalized solutions in the usual H^1 sense exist for arbitrarily small values of the positive constant h.

The new equation and some inequalities satisfied by its nonlinear terms are derived by elementary calculus. Existence is proved by modifying the arguments of Levinson (1962), as indicated herein.

Preliminaries

By H^1, H^1_0, H^2 we shall mean the usual Sobolev-Hilbert spaces over Ω, and similarly for L^p. The boundary of Ω is assumed to be smooth enough for the compact imbedding $H^1 \to L^p$ (all finite $p \geqslant 1$) to hold. (see Adams 1975, Sobolev 1963).

It follows by elementary computations that for a twice differentiable function Θ, eqn. (1) is equivalent to

$$\Delta\left(\Theta + \frac{h}{4}\,\xi^2(1-\mu^2)\,\Theta^N\right) = -\left(1+\frac{h}{2}\right)\Theta^N.$$

By a 'generalized solution' of (1) and (2) we shall understand a function Θ belonging to L^{2N} such that

$$\Theta + \frac{h}{4}\,\xi^2(1-\mu^2)\,\Theta^N$$

belongs to H^1_0, and the equivalent equation above holds in the H^1 sense.

On H^1_0 we shall use the equivalent norm given by

$$\| u \|^2 = \Sigma \iint_\Omega (Du)^2$$

where the summation is taken over all first order derivatives D.

The Result

Theorem — For a plane bounded domain Ω satisfying the conditions explained above, there exist non-trivial generalized solutions of (1) and (2) with $N = 2$ and arbitrarily small $h > 0$.

PROOF : Equation (1) can be written in the equivalent form

$$\Delta\left(\Theta + \frac{h}{4}\,\xi^2(1-\mu^2)\,\Theta^N\right) = -\left(1+\frac{h}{2}\right)\Theta^N.$$

We substitute

$$\Phi = \Theta + \frac{h}{4}\,\xi^2(1-\mu^2)\,\Theta^N$$

$$= \Theta + \frac{h}{4}\,\xi^2(1-\mu^2)\,\Theta^2$$

and express Θ in terms of Φ as

$$\Theta = [-1 + \{1 + h\xi^2(1-\mu^2)\,\Phi\}^{1/2}] \Big/ \frac{h\xi^2}{2}(1-\mu^2).$$

Here we have ignored the possibility of a negative sign before the radical, because on any part of the domain where the negative sign holds, Θ has to stay away from 0 and (2) cannot be satisfied. The equation to be satisfied by Φ is now seen to be

$$\Delta\Phi = -\left(1+\frac{h}{2}\right)\frac{2 + h\xi^2(1-\mu^2)\,\Phi - 2\,\{1 + h\xi^2(1-\mu^2)\,\Phi\}^{1/2}}{\frac{1}{4}h^2\xi^4(1-\mu^2)^2}.$$

Writing $\Psi = h\Phi$, we get the equivalent boundary value problem for Ψ

$$\Delta\Psi = -\frac{1+\frac{1}{2}h}{\frac{1}{4}h}\,\frac{2 + \xi^2(1-\mu^2)\,\Psi - 2\,\{1 + \xi^2(1-\mu^2)\,\Psi\}^{1/2}}{\xi^4(1-\mu^2)^2} \qquad \text{...(4)}$$

$$\Psi\ (\text{boundary}) = 0. \qquad \text{...(5)}$$

The right side of eqn. (4) appears to have a singularity at the origin, but rationalization leads to the equation

$$\Delta\Psi = -\frac{1+\frac{1}{2}h}{\frac{1}{4}h}\,\frac{\Psi^2}{(1+\sqrt{1+\xi^2(1-\mu^2)\,\Psi})^2} \qquad \text{...(6)}$$

and now the right side is well defined at the origin. Our considerations will be for (5) and (6).

Let

$$f(\xi,\mu,\Psi) = \frac{\Psi^2}{[1+(1+\xi^2(1-\mu^2)\,\Psi)^{1/2}]^2} = \frac{[-1+(1+\xi^2(1-\mu^2)\,\Psi)^{1/2}]^2}{\xi^4(1-\mu^2)^2}.$$

Then we have

$$\frac{\partial f}{\partial\Psi} = \frac{1-(1+\xi^2(1-\mu^2)\,\Psi)^{-1/2}}{\xi^2(1-\mu^2)} \qquad \text{...(7)}$$

which is positive for $\xi^2(1-\mu^2) \neq 0$ and $\Psi > 0$. Furthermore

$$\frac{\partial^2 f}{\partial \Psi^2} = \tfrac{1}{2}(1+\xi^2(1-\mu^2)\,\Psi)^{-3/2}$$

which is also positive for $\Psi \geqslant 0$. It follows that for $0 \leqslant \Psi_1 \leqslant \Psi_2$ we have

$$\frac{\partial f}{\partial \Psi}(\Psi_1) \leqslant \frac{\partial f}{\partial \Psi}(\Psi_2)$$

and hence

$$f(\xi, \mu, \Psi_2) - f(\xi, \mu, \Psi_1) \leqslant (\Psi_2 - \Psi_1)\frac{\partial f}{\partial \Psi}(\Psi_2).$$

Combining this with (7) and the elementary inequality

$$\frac{1-(1+a\Psi)^{-1/2}}{a} < \frac{\Psi}{2} \qquad \text{for} \quad a > 0,\ \Psi > 0,$$

we have

$$f(\xi, \mu, \Psi_2) - f(\xi, \mu, \Psi_1) \leqslant (\Psi_2 - \Psi_1)\frac{\Psi_2}{2}.$$

It can be directly verified that this holds even if $\xi^2(1-\mu^2) = 0$. Thus for non-negative Ψ_1 and Ψ_2 we have

$$|\, f(\xi, \mu, \Psi_2) - f(\xi, \mu, \Psi_1)\,| \leqslant |\, \Psi_2 - \Psi_1\,| \frac{\max\{\Psi_1, \Psi_2\}}{2}. \qquad \ldots(8)$$

Define the function F by

$$F(\xi, \mu, \Psi) = \frac{1}{\xi^4(1-\mu^2)^2}\left[2\Psi - \frac{4}{3\xi^2(1-\mu^2)}(1+\xi^2(1-\mu^2)\,\Psi)^{3/2} + \frac{4}{3\xi^2(1-\mu^2)} + \frac{\xi^2(1-\mu^2)\,\Psi^2}{2}\right].$$

This function is undefined if $\xi^2(1-\mu^2) = 0$. However, its limit as $\xi^2(1-\mu^2) \to 0$ can be seen to be $\Psi^3/12$ by using the Taylor expansion

$$(1+t)^{3/2} = 1 + \frac{3t}{2} + \frac{3t^2}{8} - \frac{1}{16}t^3(1+T)^{-3/2}, \quad 0 < t < T. \qquad \ldots(9)$$

So we understand $F(\xi, \mu, \Psi)$ to mean $\Psi^3/12$ when $\xi^2(1-\mu^2) = 0$, and it is now easy to calculate the partial derivative $\partial F/\partial \Psi$ and verify that

$$\frac{\partial F}{\partial \Psi} = f \qquad \text{and} \qquad F(\xi, \mu, 0) = 0.$$

Let $\tau > 0$ be such that $\iint F(\xi, \mu, \psi) = \tau$ for some $\psi \in H_0^1$ and $\mathcal{F}$ be the class of all $\psi \in H_0^1$ for which this equality holds. Following Levinson (1962), where the letter 'μ' is used in place of our 'τ', we get

$$m(\tau) = \inf\{\|\psi\| : \psi \in \mathcal{F}\}.$$

Let φ_j be a sequence in $\mathcal{F}$ such that $\|\varphi_j\| \to m(\tau)$. By the Sobolev-Kondrashev imbedding $H^1 \to L^4$, there is a subsequence φ_j' which converges in L^4 and hence in L^2. A modification of the argument of Levison (1962, Lemma 2.2) shows that there exists $\alpha(\tau) > 0$ such that

$$\lim \int\int \varphi_j' \cdot f(\xi, \mu, \varphi_j') = \alpha(\tau)$$

and that φ_j' converges in H_0^1 to a generalized solution of

$$\Delta\Psi = -\lambda f(\xi, \mu, \Psi), \quad \text{where} \quad \lambda = m(\tau)/\alpha(\tau). \qquad \text{...(10)}$$

The major modification required is for coping with the non-availability of the Lipschitz condition used by Levinson. To do this, we redefine f and F for $\Psi < 0$ so as to make f odd and F even, as Levinson also does. Then (8) takes the form

$$|f(\xi, \mu, \Psi_2) - f(\xi, \mu, \Psi_1)| \leqslant |\Psi_2 - \Psi_1| \cdot \frac{\max\{|\Psi_1|, |\Psi_2|\}}{2}.$$

This combined with the fact that max $\{\varphi_j', \varphi_k'\}$ is bounded in L^2-norm independently of j and k enables us to avoid the Lipschitz condition.

This establishes the existence of a solution of (5) and (6) for some value of h. Our objective however is to show that solutions can be found for arbitrarily small $h > 0$, i.e. for arbitrarily large λ in (10). We begin by proving an inequality involving the functions f and F.

The Taylor expansion (9) shows that for $t > 0$ and some $T \in (0, t)$, we have

$$-\tfrac{4}{3}(1 + t)^{3/2} + \tfrac{4}{3} + 2t + \frac{t^2}{2} = \frac{t^3}{12}(1 + T)^{-3/2}$$
$$\geqslant \frac{t^3}{12}(1 + t)^{-3/2}.$$

Therefore

$$\frac{t^3}{[1 + (1 + t)^{1/2}]^4} \leqslant \frac{12}{16}(1 + t)^{3/2}[-\tfrac{4}{3}(1 + t)^{3/2} + \tfrac{4}{3} + 2t + \tfrac{1}{2}t^2].$$

If we multiply both sides by t, substitute $t = \xi^2(1 - \mu^2)\Psi$ and then divide both sides by $[\xi^2(1 - \mu^2)]^4$, the resulting inequality reads

$$f(\xi, \mu, \Psi)^2 \leqslant \tfrac{3}{4}(1 + \xi^2(1 - \mu^2)\Psi)^{3/2} \cdot F(\xi, \mu, \Psi)\Psi \qquad \text{...(11)}$$

which is valid for $\Psi \geqslant 0$. Since the solutions of (10) which we get are nonnegative, it is permissible to substitute a solution $\Psi(\xi, \mu)$ in (11), as we shall be doing below.

The same Taylor expansion also shows that $F(\xi, \mu, \Psi) \leqslant \Psi^3/12$ for $\Psi \geqslant 0$. Since F is redefined to be an even function of Ψ, we have $F(\xi, \mu, \Psi) \leqslant |\Psi|^3/12$.

This shows that if ψ belongs to the class $\mathcal{F}$ corresponding to some $\tau > 0$, then there exist arbitrarily small positive $\tau' < \tau$ and $\epsilon < 1$ such that $\epsilon\psi \in \mathcal{F}'$. Since

$$\| \epsilon\psi \| = \epsilon \| \psi \| < \| \psi \|$$

we have $m(\tau') \leqslant \epsilon^2 m(\tau) < m(\tau)$. Thus, starting with a τ which is admissible (i.e. $\mathcal{F} \neq \phi$), it is possible to find arbitrarily small positive τ' such that $m(\tau') < m(\tau)$. Since the solutions of (10) obtained above satisfy $\| \Psi \|^2 = m(\tau)$, it follows from the imbedding $H^1 \to L^6$ that it is possible to have solutions corresponding to arbitrarily small τ but all having a common upper bound on their L^6 and L^4 norms.

We now argue as in Levinson (1962, p. 269). Letting C_1 be the constant of the imbedding $H^1 \to L^4$, we have for any admissible τ

$$m(\tau) \geqslant \frac{1}{C_1^2} \left(\iint \Psi^4\right)^{1/2}. \qquad \text{...(12)}$$

On the other hand, it follows from (11) and the Hölder inequality that

$$\begin{aligned}\alpha(\tau) &= \iint \Psi \cdot f(\xi, \mu, \Psi) \leqslant \iint \frac{\sqrt{3}}{2} \Psi^{3/2} (1 + \xi^2(1-\mu^2)\Psi)^{3/4} F(\xi, \mu, \Psi)^{1/2} \\ &\leqslant \frac{\sqrt{3}}{2} \left(\iint \Psi^3 (1 + \xi^2(1-\mu^2)\Psi)^{3/2}\right)^{1/2} \cdot \left(\iint F(\xi, \mu, \Psi)\right)^{1/2} \\ &\leqslant \frac{\sqrt{3}}{2} \tau^{1/2} \left(\iint \Psi^4\right)^{3/8} \left(\iint (1 + \xi^2(1-\mu^2)\Psi)^6\right)^{1/8}.\end{aligned}$$

From this inequality and (12), it follows that

$$\begin{aligned}\lambda = \lambda(\tau) = \frac{m(\tau)}{\alpha(\tau)} &\geqslant \frac{2C_1^2}{\sqrt{3}} \tau^{-1/2} \left(\iint \Psi^4\right)^{1/8} \left(\iint (1 + \xi^2(1-\mu^2)\Psi)^6\right)^{-1/8} \\ &\geqslant C_2 \tau^{-1/2} \left(\iint \Psi^4\right)^{1/8} \qquad \text{...(13)}\end{aligned}$$

where the constant $C_2 > 0$ can be chosen so as to be valid for arbitrarily small values of τ in view of the preceding paragraph.

Now there exists a constant $C_3 > 0$ such that $(\iint \psi^2)^{1/2} \leqslant C_3 \| \psi \|$ for $\psi \in H_0^1$. Then the H_0^1-solution of $\Delta\psi = \psi_0$ satisfies $\| \psi \| \leqslant C_3 (\iint \psi_0^2)^{1/2}$ and hence

$$\left(\iint \psi^4\right)^{1/2} \leqslant C_1^2 C_3^2 \iint \psi_0^2 = C_1^2 C_3^2 \iint (\Delta\psi)^2.$$

Also, f satisfies the inequality $| f(\xi, \mu, \psi) | \leqslant \psi^2/4$ for all ψ, as is easy to check. Therefore the solution Ψ of (10) satisfies

$$\begin{aligned}\left(\iint \Psi^4\right)^{1/2} &\leqslant C_1^2 C_3^2 \left(\iint (\Delta\Psi)^2\right) \leqslant C_1^2 C_3^2 \lambda^2 \iint f(\xi, \mu, \Psi)^2 \\ &\leqslant C_1^2 C_3^2 \frac{\lambda^2}{16} \iint \Psi^4,\end{aligned}$$

$$\left(\iint \Psi^4\right)^{1/2} \geqslant 16 C_1^{-2} C_3^{-2} \lambda^{-2}.$$

Using (13), we now get

$$\lambda \geqslant C_2 \,.\, \tau^{-1/2} \,.\, 2(C_1 C_3)^{-1/2}\, \lambda^{-1/2}$$

or

$$\lambda^{3/2} \geqslant 2C_2(C_1 C_3)^{-1/2}\, \tau^{-1/2}.$$

Since the constants C_1, C_3 depend only on the domain Ω and C_2 can be chosen so as to be valid for arbitrarily small $\tau > 0$, it follows from the above inequality that λ can be made arbitrarily large by choosing τ sufficiently small.

From (8), it follows that $|\, f(\xi, \mu, \Psi)\,| \leqslant \Psi^2/2$. Therefore if Ψ is a solution of (5 and 6), then $f(\xi, \mu, \Psi)$ lies in L^2. It is now easy to see that if Θ is defined in terms of Ψ via Φ as indicated at the beginning of the proof, then Θ is a generalized solution of (1) and (2) in the sense understood here.

ACKNOWLEDGEMENT

The author wishes to thank Professor S. K. Trehan for drawing his attention to the problem.

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