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THERMAL STRESSES IN AN INFINITE CYLINDER

HARINDER SINGH

Department of Mathematics, Guru Nanak Dev University, Amritsar 143005

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The problem of dynamic thermal stresses in an infinite isotropic elastic cylinder of radius R, with its axis along the z-axis due to the temperature boundary condition

$$T = 0 \quad \text{for} \quad z > \nu t$$
$$= T_0 \quad \text{for} \quad 0 \leqslant z \leqslant \nu t$$
$$= 0 \quad \text{for} \quad z < 0$$

at $r = R$; where ν is a positive constant and t the time. This type of situation can arise due to melting at constant rate of an insulating material at zero temperature deposited on the positive half of the infinite cylinder. The heat conduction equation has been solved by the Fourier transform method and by applying the theory of complex variables. The final results for temperature, the displacement components and the stress components have been obtained in the form of series involving Bessel function of first kind and of order zero, and its zeros.

1. INTRODUCTION

Dynamic thermal stresses in infinite cylinders have been discussed by many research workers. Dhaliwal and Choudhary (1968) employed methods of integral transforms and variation of parameters to solve dynamic thermoelastic problem for cylindrical regions. Wadhawan (1971) solved the dynamic problem of thermal stresses in an infinite cylinder when its lateral surface is held rigidly fixed and surface temperature is a function of time. He used the Laplace transform to solve the problem.

The present paper deals with the dynamic thermal stresses in an infinite cylinder of radius R due to a constant temperature applied to a variable portion of the curved surface while the rest of surface is maintained at zero temperature. Such situation can arise due to melting of insulating material deposited on the surface of the cylinder.

The heat conduction equation has been solved by applying the Fourier transform and the theory of complex variable. The thermoelastic equation of motion has been split into two wave equations representing rotational and irrotational displacement. The particular integral of the longitudinal wave equation has been obtained by the method given by Singh and Puri (1963). The final results have been obtained in the form of series involving Bessel function of first kind and of order zero, and its zeros.

2. Thermoelastic Equations

We consider the problem of thermoelastic stresses in an isotropic infinite cylinder of radius R, whose axis is along the z-axis, under the temperature boundary condition

$$\left.\begin{aligned} T &= 0 \quad \text{for} \quad z > \nu t \\ &= T_0 \quad \text{for} \quad 0 \leqslant z \leqslant \nu t \\ &= 0 \quad \text{for} \quad z < 0 \end{aligned}\right\} \qquad \ldots(2.1)$$

at $r = R$ and $t > 0$; where t is time and ν a positive constant. The condition of form (2.1) can arise due to melting, at a constant rate of an insulating material at zero temperature deposited on the positive half of the infinite cylinder.

The curved surface of the cylinder is assumed to be stress free i.e.

$$\sigma_{rr} = \sigma_{rz} = 0 \quad \text{at} \quad r = R. \qquad \ldots(2.2)$$

It is also assumed that

$$T = \frac{\partial T}{\partial z} = 0 \quad \text{as} \quad z \to \pm \infty. \qquad \ldots(2.3)$$

Due to symmetry about z-axis the temperature, the displacement and the stresses are independent of the coordinate ϕ.

Introducing the non-dimensional quantities

$$r' = r/R;\ z' = z/R;\ T' = T/T_0;\ \vec{u'} = \vec{u}/R;\ t' = t/R$$

the heat conduction equation and elastic equation of motion can be written in the form

$$\left.\begin{aligned} \nabla'^2 T' &= \frac{\rho_0 c_e}{k} \frac{\partial T'}{\partial t'} \\ &= \frac{1}{k} \frac{\partial T'}{\partial t'} \end{aligned}\right\} \qquad \ldots(2.4)$$

$$(\lambda + \mu)\, \nabla'(\nabla' . \vec{u'}) + \mu \nabla'^2 \vec{u'} = (3\lambda + 2\mu)\, \alpha R T_0 \nabla' T' + \rho_0 \nu^2 \frac{\partial^2 \vec{u'}}{\partial t'^2} \qquad \ldots(2.5)$$

where k is coefficient of thermal conductivity; ρ_0 the density; c_e the specific heat; α the coefficient of thermal expansion and λ, μ are Lame's constants.

The boundary condition (2.1) in the non-dimensional form is

$$\left.\begin{aligned} T'(1, z, t) &= 0 \quad \text{for} \quad z' > t' \\ &= 1 \quad \text{for} \quad 0 \leqslant z' \leqslant t' \\ &= 0 \quad \text{for} \quad z' < 0. \end{aligned}\right\} \qquad \ldots(2.6)$$

In the following discussion we neglect the primes.

3. Solution of the Thermal Problem

Applying the Fourier transform

$$\bar{X}(p) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} X(z)\, e^{ipz}\, dz, \quad i^2 = -1$$

to eqns. (2.4) and (2.6) and using (2.3) we get

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - p^2\right)\bar{T} = \frac{1}{k}\frac{\partial \bar{T}}{\partial t} \qquad ...(3.1)$$

and for $r = 1$; $t > 0$

$$\bar{T} = \frac{1}{ip\sqrt{(2\pi)}}(e^{ipt} - 1). \qquad ...(3.2)$$

Equation (3.1) under the boundary condition (3.2) has a solution of the form

$$\bar{T}(r, p, t) = f_1(r)\frac{e^{ipt} - 1}{ip\sqrt{(2\pi)}} + \frac{1}{ip\sqrt{(2\pi)}} f_2(r) \qquad ...(3.3)$$

where $f_1(r)$ and $f_2(r)$ satisfy the differential equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - p^2\right)[f_2(r) - f_1(r)] = 0 \qquad ...(3.4)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - p^2\right)f_1(r) = 0 \qquad ...(3.5)$$

subject to the boundary conditions

$$\left.\begin{aligned} f_2(r) - f_1(r) &= -1 \\ f_1(r) &= 1. \end{aligned}\right\} \qquad ...(3.6)$$

Equations (3.4) and (3.5) are satisfied by the solutions

$$\left.\begin{aligned} f_2(r) - f_1(r) &= A_1 I_0(pr) \\ f_1(r) &= A_2 I_0(r(p^2 + ipk^{-1})^{1/2}) \end{aligned}\right\} \qquad ...(3.7)$$

where I_0 is the modified Bessel function of order zero. Putting $r = 1$ in (3.7) and using eqns. (3.6) we get

$$A_1 = -1/I_0(p)$$

$$A_2 = 1/I_0((p^2 + ipk^{-1})^{1/2}).$$

Substituting for $f_1(r)$ and $f_2(r)$ in (3.3) we get the complete solution for (3.1) satisfying the boundary condition (3.2) to be

$$\bar{T}(r, p, t) = \frac{1}{ip\sqrt{(2\pi)}}\left[\frac{I_0(r(p^2 + ipk^{-1})^{1/2})}{I_0((p^2 + ipk^{-1})^{1/2})} e^{ipt} - \frac{I_0(pr)}{I_0(p)}\right]. \qquad ...(3.8)$$

Applying the inversion formula for the Fourier transform

$$X(z) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \bar{X}(p)\, e^{-ipz} dp$$

to (3.8) we get

$$T(r, z, t) = \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty} \frac{I_0(r(p^2 + ipk^{-1})^{1/2})}{pI_0((p^2 + ipk^{-1})^{1/2})} e^{ip(t-z)}\, dp\right.$$

$$\left. - \int_{-\infty}^{\infty} \frac{I_0(pr)}{pI_0(p)} e^{-ipz}\, dp\right]. \qquad ...(3.9)$$

The two integrals involved in (3.9) will be evaluated separately.

Evaluation of $\int_{-\infty}^{\infty} \frac{I_0(pr)}{pI_0(p)} e^{-ipz}\, dp$

Case A : *When* $z > 0$ — We consider the integral

$$\int_C \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z}\, d\lambda$$

where $\lambda = q + ip$ is a complex number with $q \geqslant 0$ and the contour

$$C = c_1 + c_2 + c_3 + \Gamma$$

is taken as shown in the Fig. 1. The contour C is denoted at the origin with a semi-circle $\Gamma : |\lambda| = \rho$, Re $(\lambda) \geqslant 0$ as $\lambda = 0$ is a pole of the integrand. The part c_1 of the contour is also a semi-circle of radius R_1.

Now $$\left|\frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z}\right| \to 0 \quad \text{as} \quad |\lambda| \to \infty$$

therefore by Jordon's lemma (Copson 1962) we have

$$\lim_{R_1 \equiv |\lambda| \to \infty} \int_{C_1} \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z}\, d\lambda = 0. \qquad ...(3.10)$$

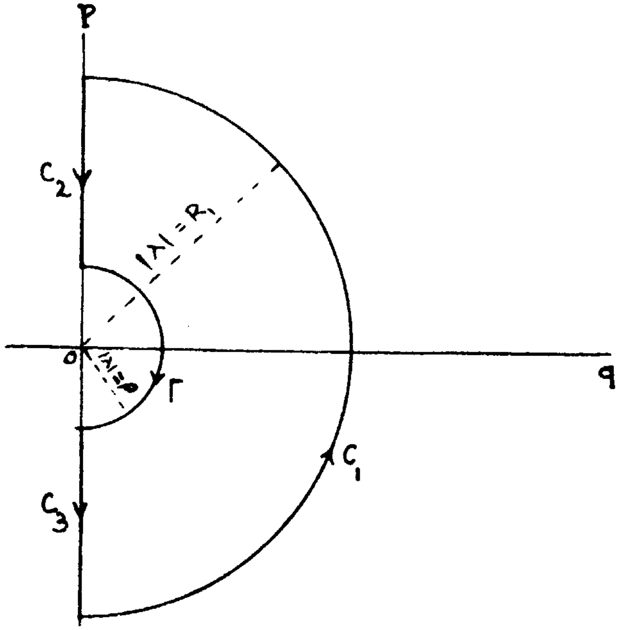

FIG. 1.

Also we have

$$\lim_{\rho\to 0} \int_{\Gamma} \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z} d\lambda = -\tfrac{1}{2} \,.\, 2\pi i \text{ Res} \left\{\frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z}\right\}_{\lambda=0}$$

$$= -\pi i \qquad \ldots(3.11)$$

and

$$\lim_{\substack{R_1\to\infty \\ \rho\to\infty}} \left[\int_{C_2} \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z} d\lambda + \int_{C_3} \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z} d\lambda \right]$$

$$= - \int_{-\infty}^{\infty} \frac{I_0(pr)}{pI_0(p)} e^{-ipz} dp. \qquad \ldots(3.12)$$

Using (3.10)–(3.12) in the Residue theorem

$$\lim_{\substack{R_1\to\infty \\ \rho\to 0}} \int_{C} \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z} d\lambda = 2\pi i \sum_{n=1}^{\infty} \text{Res} \left\{\frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z}\right\}_{\lambda=\alpha_n}$$

where α_n are positive zeros of $J_0(\alpha) = 0$ arranged in the increasing order of their magnitudes, we get

$$\int_{-\infty}^{\infty} \frac{I_0(pr)}{pI_0(p)} e^{-ipz}\, dp = -\pi i + 2\pi i \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} e^{-\alpha_n z}. \qquad \ldots(3.13)$$

Case B : *When* $z < 0$ — We consider the integral

$$\int_C \frac{J_0(\lambda r)}{\lambda J_0(\lambda)} e^{-\lambda z}\, d\lambda$$

where $\lambda = q + ip$; $q \leqslant 0$ and C is the contour as shown in Fig. 2.

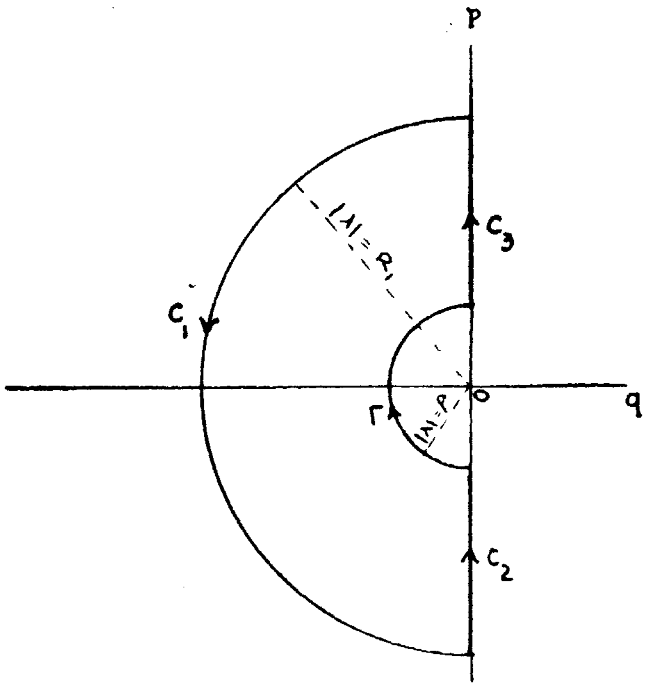

FIG. 2.

Proceeding as before we get

$$\int_{-\infty}^{\infty} \frac{I_0(pr)}{pI_0(p)} e^{-ipz}\, dp = \pi i - 2\pi i \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp(\alpha_n z). \qquad \ldots(3.14)$$

Evaluation of $\displaystyle\int_{-\infty}^{\infty} \frac{I_0(r(p^2 + ipk^{-1})^{1/2})}{pI_0((p^2 + ipk^{-1})^{1/2})} e^{ip(t-z)}\, dp$

Case A : *When* $z > 0$ — Now two cases arise according as $t - z$ is negative or positive. When $t - z$ is negative we consider the integral

$$\int_C \frac{J_0(r(\lambda^2 - \lambda k^{-1})^{1/2})}{\lambda J_0((\lambda^2 - \lambda k^{-1})^{1/2})} e^{\lambda(t-z)}\, d\lambda$$

where $\lambda = q + ip$; $q \geqslant 0$ and the contour C is as shown in Fig. 1. Applying Jordon's lemma and the residue theorem it is found that

$$\int_{-\infty}^{\infty} \frac{I_0(r(p^2 + ipk^{-1})^{1/2})}{pI_0((p^2 + ipk^{-1})^{1/2})} e^{ip(t-z)}\, dp$$

$$= -\pi i + 2\pi i \sum_{n=1}^{\infty} \frac{2\alpha_n J_0(r\alpha_n)}{\eta_n(2\eta_n - k^{-1})\, J_1(\alpha_n)} \exp\,(\eta_n(t - z) \qquad \ldots(3.15)$$

where $\qquad \eta_n = (1 + (1 + 4k^2\alpha_n^2)^{1/2})/2k;\ n \geqslant 1.$

When $t - z$ is positive we consider the integral

$$\int_C \frac{J_0(r(\lambda^2 - \lambda k^{-1})^{1/2})}{\lambda J_0((\lambda^2 - \lambda k^{-1})^{1/2})} e^{\lambda(t-z)}\, d\lambda$$

where $\lambda = q + ip$; $q \leqslant 0$ and contour C is taken as in Fig. 2. Proceeding as before we get

$$\int_{-\infty}^{\infty} \frac{I_0(r(p^2 + ipk^{-1})^{1/2})}{pI_0((p^2 + ipk^{-1})^{1/2})} e^{ip(t-z)}\, dp$$

$$= \pi i - 2\pi i \sum_{n=1}^{\infty} \frac{2\alpha_n J_0(r\alpha_n)}{\mu_n(2\mu_n - k^{-1})\, J_1(\alpha_n)} \exp\,(\mu_n(t - z)) \qquad \ldots(3.16)$$

where $\qquad \mu_n = (1 - (1 + 4k^2\alpha_n^2)^{1/2})/2k;\ n \geqslant 1.$

Case B : *When* $z < 0$ — In this case $t - z$ is positive and therefore the result (3.16) also holds for the case when $z < 0$.

Using (3.13) to (3.16) in (3.9), we get

$$T(r, z, t) = \sum_{n=1}^{\infty} \frac{2\alpha_n J_0(r\alpha_n)}{\eta_n(2\eta_n - k^{-1})\, J_1(\alpha_n)} \exp\,(\eta_n(t - z))$$

$$- \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp\,(-\alpha_n z) \qquad \text{for} \quad z > t$$

(*equation continued on p.* 412)

$$= 1 - \sum_{n=1}^{\infty} \frac{2\alpha_n J_0(r\alpha_n)}{\mu_n(2\mu_n - k^{-1}) J_1(\alpha_n)} \exp(\mu_n(t - z))$$

$$- \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp(-\alpha_n z) \quad \text{for} \quad 0 \leqslant z \leqslant t$$

$$= \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp(\alpha_n z) - \sum_{n=1}^{\infty} \frac{2\alpha_n J_0(r\alpha_n)}{\mu_n(2\mu_n - k^{-1}) J_1(\alpha_n)}$$

$$\times \exp(\mu_n(t - z)) \quad \text{for} \quad z < 0. \qquad \ldots(3.17)$$

4. Elastic Problem

Taking the displacemet $\vec{u}$ in the form

$$\vec{u} = \nabla \Phi + \nabla \times \vec{\Psi}$$

in the equation of motion (2.5), we get two wave equations

$$\nabla^2 \Phi - \frac{1}{c_1^2} \frac{\partial^2 \Phi}{\partial t^2} = mT \qquad \ldots(4.1)$$

$$\nabla^2 \vec{\Psi} - \frac{1}{c_2^2} \frac{\partial^2 \vec{\Psi}}{\partial t^2} = 0 \qquad \ldots(4.2)$$

where

$$m = \frac{3\lambda + 2\mu}{\lambda + 2\mu} \alpha R T_0$$

$$c_1^2 = \frac{\lambda + 2\mu}{\rho_0 \nu^2}; \quad c_2^2 = \frac{\mu}{\rho_0 \nu^2}.$$

Here the scalar Φ and the vector $\vec{\Psi}$ represent irrotational and rotational parts of the displacement $\vec{u}$. The cylinder being bounded by the curved surface, the stress distribution includes the effect of both Φ and $\vec{\Psi}$. It is possible to take only one component of the vector $\vec{\Psi}$ to be non-zero i.e. We can take

$$\vec{\Psi} \equiv (0, \psi_{,r}, 0)$$

where $\psi_{,r} = \dfrac{\partial \psi}{\partial r}$ and ψ satisfies the wave eqn. (4.2) and therefore has the general form

$$\psi = \sum_{n=1}^{\infty} D_n J_0(r(p_n^2 - q_n^2 c_2^{-2})^{1/2}) \exp(-p_n z - q_n t) \quad ...(4.3)$$

where D_n, p_n, q_n are constants to be determined from the boundary conditions.

To obtain the particular integral of (4.1), we use the method given by Singh and Puri (1963). Eliminating T from (2.4) and (4.1), we get

$$\left(\nabla^2 - \frac{1}{k}\frac{\partial}{\partial t}\right)\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi = 0.$$

We take $\Phi = \Phi_1 + \Phi_2$ where Φ_1 and Φ_2 satisfy the equations

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi_1 = 0 \quad ...(4.4)$$

$$\left(\nabla^2 - \frac{1}{k}\frac{\partial}{\partial t}\right)\Phi_2 = 0. \quad ...(4.5)$$

Also Φ is the solution of (4.1) therefore

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi_2 = mT. \quad ...(4.6)$$

Now to determine Φ_2, the particular integral of (4.1), subtracting (4.6) from (4.5), we get

$$\left(\frac{\partial^2}{\partial t^2} - b^2\frac{\partial}{\partial t}\right)\Phi_2 = -mc_1^2 T \quad ...(4.7)$$

where $b^2 = c_1^2/k$.

Integrating the differential eqn. (4.7), we get

$$\Phi_2 = F_1(r, z) + F_2(r, z)\, e^{b^2 t} - \frac{mc_1^2}{b^2}\left[e^{b^2 t}\int e^{-b^2 t} T\, dt - \int T\, dt\right]. \quad ...(4.8)$$

Since $b^2 > 0$, to satisfy the physical conditions we must take

$$F_2(r, z) = 0.$$

Now Φ_2 being a solution of (4.5), substituting (4.8) in (4.5), we get $\nabla^2 F_1 = 0$. Thus F_1 has the general form

$$F_1(r, z) = \sum_{n=1}^{\infty} A_n J_0(\beta_n r) \exp(-\beta_n z) \quad ...(4.9)$$

where A_n and β_n are arbitrary constants.

We take Φ_1, the solution of (4.4), in the form

$$\Phi_1 = \sum_{n=1}^{\infty} B_n J_0(r(e_n^2 - d_n^2 c_1^{-2})^{1/2}) \exp(-e_n z - d_n t) + t \sum_{n=1}^{\infty} b_n J_0(r\delta_n) \exp(-\delta_n z). \qquad \ldots(4.10)$$

where B_n, e_n, d_n, b_n and δ_n are arbitrary constants.

Combining (4.8), (4.9) and (4.10) we get the complete solution of (4.1) to be

$$\Phi = \sum_{n=1}^{\infty} B_n J_0(r(e_n^2 - d_n^2 c_1^{-2})^{1/2}) \exp(-e_n z - d_n t) + t \sum_{n=1}^{\infty} b_n J_0(r\delta_n) \exp(-\delta\eta z) + \sum_{n=1}^{\infty} A_n J_0(\beta_n r) \exp(-\beta_n z) - \frac{mc_1^2}{b^2}\left[\exp(b^2 t)\int \exp(-b^2 t)\, T\, dt - \int T\, dt\right]. \qquad \ldots(4.11)$$

The displacement components in terms of Φ and ψ are

$$\left.\begin{aligned} u_r &= \Phi_{,r} - \psi_{,rz} \\ u_z &= \Phi_{,z} - \psi_{,zz} + \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \\ u_\phi &= 0. \end{aligned}\right\} \qquad \ldots(4.12)$$

Using the stress-strain relations,

$$\sigma_{ij} = 2\mu e_{ij} + \lambda\delta_{ij} e_{kk}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) - \alpha\delta_{ij} T.$$

We get

$$\left.\begin{aligned} \frac{1}{2\mu}\sigma_{rr} &= \Phi_{,rr} - \nabla^2\Phi + \frac{1}{2c_2^2}\frac{\partial^2\Phi}{\partial t^2} - \psi_{,rrz} \\ \frac{1}{2\mu}\sigma_{rz} &= \Phi_{,rz} - \psi_{,rzz} + \frac{1}{2c_2^2}\frac{\partial^2\psi_{,r}}{\partial t^2} \\ \frac{1}{2\mu}\sigma_{\phi\phi} &= \frac{1}{r}\Phi_{,r} - \nabla^2\Phi + \frac{1}{2c_2^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{r}\psi_{,rz} \\ \frac{1}{2\mu}\sigma_{zz} &= \Phi_{,zz} - \nabla^2\Phi + \frac{1}{2c_2^2}\frac{\partial^2\Phi}{\partial t^2} - \psi_{,zzz} + \frac{1}{c_2^2}\frac{\partial^2\psi_{,z}}{\partial t^2} \\ \sigma_{\phi z} &= \sigma_{r\phi} = 0. \end{aligned}\right\} \qquad \ldots(4.13)$$

To determine the arbitrary constants involved in (4.3) and (4.11), we shall consider three cases :

Case A : $z > t$

Case B : $0 \leqslant z \leqslant t$

Case C : $z < 0$

as we have different expressions for T for the three ranges of z.

Case A

In this case we have from (4.11) and (3.17)

$$\Phi = \sum_{n=1}^{\infty} B_n J_0(r(e_n^2 - d_n^2 c_1^{-2})^{1/2}) \exp(-e_n z - d_n t)$$

$$+ t \sum_{n=1}^{\infty} b_n J_0(r\delta_n) \exp(-\delta_n z) + \sum_{n=1}^{\infty} A_n J_0(r\beta_n) \exp(-\beta_n z)$$

$$- mc_1^2 \sum_{n=1}^{\infty} \frac{2\alpha_n J_0(r\alpha_n)}{\eta_n(2\eta_n - k^{-1})(\eta_n - b^2) J_1(\alpha_n)} \exp(\eta_n(t - z))$$

$$- mc_1^2 \frac{1 + b^2 t}{b^4} \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp(-\alpha_n z). \qquad \ldots(4.14)$$

Substituting for Φ from (4.14) and for ψ from (4.3) in the expressions for stress components σ_{rr} and σ_{rz} as given by (4.13), and applying the boundary conditions (2.2), we get

$$\sum_{n=1}^{\infty} B_n [(e_n^2 - d_n^2 c_1^{-2}) J_2((e_n^2 - d_n^2 c_1^{-2})^{1/2})$$

$$- (e_n^2 + d_n^2 c_1^{-2} - d_n^2 c_2^{-2}) J_0((e_n^2 - d_n^2 c_1^{-2})^{1/2})] \exp(-e_n z - d_n t)$$

$$+ t \sum_{n=1}^{\infty} b_n \delta_n^2 [J_2(\delta_n) - J_0(\delta_n)] \exp(-\delta_n z)$$

$$+ \sum_{n=1}^{\infty} A_n \beta_n^2 [J_2(\beta_n) - J_0(\beta_n)] \exp(-\beta_n z)$$

$$+ \sum_{n=1}^{\infty} D_n p_n (p_n^2 - q_n^2 c_2^{-2}) [J_2((p_n^2 - q_n^2 c_2^{-2})^{1/2})$$

$$- J_0((p_n^2 - q_n^2 c_2^{-2})^{1/2})] \exp(-p_n z - q_n t) -$$

(*equation continued on p.* 416)

$$-2mc_1^2\sum_{n=1}^{\infty}\frac{\alpha_n^3 J_2(\alpha_n)}{\eta_n^2(2\eta_n-k^{-1})(\eta_n-b^2)J_1(\alpha_n)}\exp(\eta_n(t-z))$$

$$-mc_1^2\frac{1+b^2t}{b^4}\sum_{n=1}^{\infty}\frac{\alpha_n J_2(\alpha_n)}{J_1(\alpha_n)}\exp(-\alpha_n z)=0 \qquad \ldots(4.15)$$

$$\sum_{n=1}^{\infty} B_n e_n(e_n^2-d_n^2c_1^{-2})[J_2((e_n^2-d_n^2c_1^{-2})^{1/2})$$

$$+J_0((e_n^2-d_n^2c_1^{-2})^{1/2})]\exp(-e_nz-d_nt)$$

$$+t\sum_{n=1}^{\infty} b_n\delta_n^3[J_2(\delta_n)+J_0(\delta_n)]\exp(-\delta_nz)$$

$$+\sum_{n=1}^{\infty} A_n\beta_n^3[J_2(\beta_n)+J_0(\beta_n)]\exp(-\beta_nz)$$

$$+\sum_{n=1}^{\infty} D_n\left(p_n^2-\frac{q_n^2}{2c_2^2}\right)(p_n^2-q_n^2c_2^{-2})[J_2((p_n^2-q_n^2c_2^{-2})^{1/2})$$

$$+J_0((p_n^2-q_n^2c_2^{-2})^{1/2})]\exp(-p_nz-q_nt)$$

$$-2mc_1^2\sum_{n=1}^{\infty}\frac{\alpha_n^3 J_2(\alpha_n)}{\eta_n(2\eta_n-k^{-1})(\eta_n-b^2)J_1(\alpha_n)}\exp(\eta_n(t-z))$$

$$-mc_1^2\frac{1+b^2t}{b^4}\sum_{n=1}^{\infty}\frac{\alpha_n^2 J_2(\alpha_n)}{J_1(\alpha_n)}\exp(-\alpha_nz)=0.$$

Taking $e_n=p_n=\eta_n$, $d_n=q_n=-\eta_n$, $\delta_n=\beta_n=\alpha_n$ and equating the coefficients of $\exp(-\alpha_nz)$, $t\exp(-\alpha_nz)$ and $\exp(\eta_n(t-z))$ to zero we get

$$A_n=\frac{mc_1^2}{b^4\alpha_nJ_1(\alpha_n)}$$

$$b_n=\frac{mc_1^2}{b^2\alpha_nJ_1(\alpha_n)}$$

$$B_n=2mc_1^2(1-c_2^{-2})\,\Delta\,\eta_n\Bigg[\left(2-\frac{1}{2c_2^2}\right)J_0(\eta_n(1-c_2^{-2})^{1/2})$$

$$-\frac{1}{2c_2^2}J_2(\eta_n(1-c_2^{-2})^{1/2})\Bigg]$$

$$D_n = -2mc_1^2(2 - c_2^{-2}) \triangle J_0(\eta_n(1 - c_1^{-2})^{1/2})$$

where

$$\triangle = \alpha_n^3 J_2(\alpha_n)\Big[(1 - c_2^{-2})\,\eta_n^5\,(2\eta_n - k^{-1})\,(\eta_n - b^2)\,J_1(\alpha_n)$$

$$\times\Big\{(1 - \tfrac{1}{2}c_2^{-2})\,[J_0(\eta_n(1 - c_2^{-2})^{1/2}) + J_2(\eta_n(1 - c_2^{-2})^{1/2})]$$

$$\times [(1 - c_1^{-2})\,J_2(\eta_n(1 - c_1^{-2})^{1/2}) - (1 + c_1^{-2} - c_2^{-2})$$

$$\times J_0(\eta_n(1 - c_1^{-2})^{1/2})] - (1 - c_1^{-2})\,[J_2(\eta_n(1 - c_2^{-2})^{1/2})$$

$$- J_0(\eta_n(1 - c_2^{-2})^{1/2})]\,[J_0(\eta_n(1 - c_1^{-2})^{1/2}) + J_2(\eta_n(1 - c_1^{-2})^{1/2})]\Big\}\Big]^{-1}.$$

Substituting the values of constants given by (4.15) in (4.3) and (4.14), we get

$$\psi = 2mc_1^2 \sum_{n=1}^{\infty} D'_n J_0(r\eta_n(1 - c_2^{-2})^{1/2}) \exp(\eta_n(t - z)) \qquad \ldots(4.16)$$

$$\Phi = 2mc_1^2 \sum_{n=1}^{\infty} B'_n J_0(r\eta_n(1 - c_1^{-2})^{1/2}) \exp(\eta_n(t - z))$$

$$-2mc_1^2 \sum_{n=1}^{\infty} \frac{\alpha_n J_0(r\alpha_n)}{\eta_n^2\,(2\eta_n - k^{-1})\,(\eta_n - b^2)\,J_1(\alpha_n)} \exp(\eta_n(t - z))$$

where

$$2mc_1^2 B'_n = B_n,\quad 2mc_1^2 D'_n = D_n.$$

So that for $z > t$ the non-zero displacement and stress components are given by

$$u_r = -mc_1^2 r \sum_{n=1}^{\infty} \Big\{(1 - c_1^{-2})\,B'_n \eta_n^2\,[J_0(r\eta_n(1 - c_1^{-2})^{1/2})$$

$$+ J_2(r\eta_n(1 - c_1^{-2})^{1/2})]$$

$$+ (1 - c_2^{-2})\,D'_n\,\eta_n^3\,[J_0(r\eta_n(1 - c_2^{-2})^{1/2}) + J_2(r\eta_n(1 - c_2^{-2})^{1/2})]$$

$$- \frac{\alpha_n^3\,[J_0(r\alpha_n) + J_2(r\alpha_n)]}{\eta_n^2\,(2\eta_n - k^{-1})\,(\eta_n - b^2)\,J_1(\alpha_n)}\Big\} \exp(\eta_n(t - z)) \qquad \ldots(4.17)$$

$$\frac{1}{2\mu}\sigma_{rr} = mc_1^2 \sum_{n=1}^{\infty} \Big\{ B'_n \eta_n^2 [(1 - c_1^{-2}) J_2(r\eta_n(1 - c_1^{-2})^{1/2})$$

$$- (1 + c_1^{-2} - c_2^{-2}) J_0(r\eta_n(1 - c_1^{-2})^{1/2})]$$

$$+ (1 - c_2^{-2}) D'_n \eta_n^3 [J_2(r\eta_n(1 - c_2^{-2})^{1/2}) - J_0(r\eta_n(1 - c_2^{-2})^{1/2})]$$

$$- \frac{\alpha_n^3 [J_0(r\alpha_n) + J_2(r\alpha_n)]}{\eta_n^2 (2\eta_n - k^{-1})(\eta_n - b^2) J_1(\alpha_n)}$$

$$+ (2 - c_2^{-2}) \frac{\alpha_n J_0(r\alpha_n)}{(2\eta_n - k^{-1})(\eta_n - b^2) J_1(\alpha_n)} \Big\} \exp(\eta_n(t - z)) \qquad \ldots(4.18)$$

etc.

Adopting the same technique we can obtain the final expression in other two cases as well.

REFERENCES

Copson, E. T. (1962). An Introduction to the Theory of Functions of a Complex Variable. Oxford University Press, London.

Dhaliwal, R. S., and Choudhary, K. L. (1968). Dynamic problems of thermoelasticity for cylindrical regions. *Arch. Mech. Stos.*, **20**, 47–66.

Singh, A., and Puri, P. (1963). Dynamic thermal stresses in an infinite slab. *Arch. Mech. Stos.*, **15**, 77–88.

Wadhawan, M. C. (1971). Thermoelastic responce of isotropic media. Ph.D. Thesis, Punjabi University, Patiala, India.