Indian J. pure appl. Math., **12**(3) : 419–425, March 1981

ASYMPTOTIC ANALYSIS OF THE UNSTEADY HYDROMAGNETIC BOUNDARY LAYERS IN A ROTATING FLOW

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(*Received* 14 *March* 1980)

A study of the unsteady hydromagnetic boundary layer flow induced by the harmonic oscillation of an infinite plate in an electrically conducting rotating fluid in the presence of a uniform magnetic field is made. Induced magnetic field is also taken into consideration. Both the fluid and the plate are in a state of solid body rotation. Solutions are obtained for small and large times by means of the asymptotic behaviours of the Laplace transforms. It has been found that just after the impulsive motion is imposed on the disk and fluid, the unsteady hydromagnetic boundary layer flow is generated which consists of two distinct boundary layer thicknesses. It is shown that how steady state flow is established. It is also shown that for large time and frequency of oscillation $\omega = 0$, the associated Ekman layer has thickness lesser than classical Ekman layer.

INTRODUCTION

The importance of the effects of rotation and electromagnetic force on the hydromagnetic flow and their applications to cosmical fluid dynamics and solar physics have been drawn attention of many research workers recently. In order to discover the above effects on the hydromagnetic flow phenomena and to examine the structures of the associated boundary layers, the initial value investigation of the Stokes and the Ekman problems in the presence of a magnetic field are of considerable interest in geophysical and cosmical fluid dynamics.

Hide and Roberts (1960) have made a steady state analysis of the hydromagnetic flow induced in a viscous, incompressible rotating conducting fluid in the presence of a magnetic field by the harmonic oscillation of an infinite rigid wall. Thornely (1968) studied unsteady non-magnetic case of the above problem with rotating fluid. Debnath studied the unsteady hydrodynamic and hydromagnetic boundary layer flows in a series of papers (see Debnath 1972, 1973, 1975a, 1975b; Sen and Debnath 1978).

Our formulation is an extension of the work of Debnath. In the case of large electrical conductivity, we cannot neglect the induced magnetic field, we have to solve all the velocity field equations together with magnetic field equations, simultaneously. Being impulsive nature of problems, solutions are obtained by the method of Laplace transform. But the Laplace transform solution is very complicated and its exact inversion is extremely difficult.

In this paper solutions are obtained for small and large times by Laplace transform technique. It has been shown that conductivity of the fluid plays important role in the formation of boundary layers.

Mathematical Formulation

A semi-infinite body of an incompressible viscous and electrically conducting fluid is bounded by an infinite rigid (non-conducting) plate at $z = 0$. Both the plate and the fluid are in a state of solid body rotation with a angular velocity Ω about the z-axis normal to the plate. A uniform magnetic field $\vec{H}$ is imposed in the system parallel to z-axis. We consider the MHD flow induced in the fluid by an elliptic harmonic oscillation of the plate in its own plane.

The unsteady motion of a viscous electrically conducting fluid in the presence of a magnetic field in this rotating coordinate systems is governed by the equations

$$\frac{D\vec{u}}{Dt} + 2\Omega \times \vec{u} = -\frac{1}{\rho}\text{ grad } P + \frac{\mu_e}{\rho}\vec{J} \times \vec{H} + \nu\nabla^2\vec{u} \qquad \text{...(1)}$$

$$\frac{\partial \vec{H}}{\partial t} - \text{curl }(\vec{u} \times \vec{H}) = -\text{curl }(\eta \text{ curl } \vec{H}) \qquad \text{...(2)}$$

$$\text{div } \vec{u} = 0 \qquad \text{...(3)}$$

$$\text{div } \vec{H} = 0 \qquad \text{...(4)}$$

$$\eta = \frac{1}{\mu_e \sigma} \qquad \text{...(5)}$$

where

$\vec{u} = (u, v, w)$ is the velocity field,

$\vec{H} = (H_x, H_y, H_z)$ is the uniform magnetic field,

P is the pressure including the centrifugal term, ρ the density, η the resistivity, $\vec{J}$ the electric current density and ν the kinematic viscosity of the fluid.

Neglecting displacement currents, the Maxwell equations and the generalized Ohm's law appropriate for the problem are

$$\text{div } \vec{H} = 0, \text{ curl } \vec{H} = \vec{J}, \text{ curl } \vec{E} = -\frac{\partial \mu_e \vec{H}}{\partial t} \qquad \text{...(6)}$$

$$\vec{J} = \sigma(\vec{E} + \mu_e \vec{u} \times \vec{H}) \qquad ...(7)$$

where E is the electric field, μ_e the magnetic permeability and σ the electrical conductivity.

In addition to the basic rotation, the plate performs oscillations. We assume that

$$u + iv = a\,e^{i\omega t} + b\,e^{-i\omega t}$$

and

$$H_x + iH_y = c\,e^{i\omega t} + d\,e^{-i\omega t}$$

where a, b, c, d are complex constants and ω is the fixed frequency of oscillations.

The structure of the hydromagnetic Stokes layer in a viscous flow leads us to presume the velocity field and magnetic field in the boundary layer in the form

$$\vec{u} = (u(z, t), v(z, t), w(z, t)) \qquad ...(8)$$

$$\vec{H} = (H_x(z, t), H_y(z, t), H_z(z, t)) \qquad ...(9)$$

where

$$\left.\begin{aligned} H_z(z, t) &= B_0 = \text{const.} \\ w(z, t) &= 0. \end{aligned}\right\} \qquad ...(10)$$

The boundary conditions are taken as

$$u + iv = a\,e^{i\omega t} + b\,e^{-i\omega t}, \text{ on } z = 0, t > 0, \qquad ...(11)$$

$$H_x + iH_y = c\,e^{i\omega t} + d\,e^{-i\omega t}, \text{ on } z = 0, t > 0 \qquad ...(12)$$

and as $z \to \infty$, $t > 0$

$$u, v \to 0 \qquad ...(13)$$

and $$H_x, H_y \to 0. \qquad ...(14)$$

Equations of motion in simplified form are

$$\frac{\partial u}{\partial t} - 2v\Omega - \frac{B_0 \mu_e}{\rho}\frac{\partial H_x}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2} \qquad ...(15)$$

$$\frac{\partial v}{\partial t} + 2u\Omega - \frac{B_0 \mu_e}{\rho}\frac{\partial H_y}{\partial z} = \nu \frac{\partial^2 v}{\partial z^2} \qquad ...(16)$$

$$\frac{\partial H_x}{\partial t} - B_0 \frac{\partial u}{\partial z} = \eta \frac{\partial^2 H_x}{\partial z^2} \qquad ...(17)$$

$$\frac{\partial H_y}{\partial t} - B_0 \frac{\partial v}{\partial z} = \eta \frac{\partial H_y}{\partial z^2}. \quad \ldots(18)$$

Each pair (15) – (16) and (17) – (18) can be combined into one equation. Thus

$$\frac{\partial P}{\partial t} + 2\Omega i P - \frac{B_0 \mu_e}{\rho} \frac{\partial Q}{\partial z} = \nu \frac{\partial^2 P}{\partial z^2} \quad \ldots(19)$$

$$\frac{\partial Q}{\partial t} - B_0 \frac{\partial P}{\partial z} = \eta \frac{\partial^2 Q}{\partial z^2} \quad \ldots(20)$$

where

$$P \equiv u + iv$$

$$Q \equiv H_x + iH_y.$$

Again combining eqns. (19) and (20) we have

$$\left\{\eta\nu \frac{\partial^4}{\partial z^4} - (\eta + \nu) \frac{\partial^3}{\partial z^2 \partial t} + \frac{\partial^2}{\partial t^2} + 2\Omega i \frac{\partial}{\partial t} - (2\Omega i\eta + nB_0) \frac{\partial^2}{\partial z^2}\right\} \begin{bmatrix} P \\ Q \end{bmatrix} = 0. \quad \ldots(21)$$

This equation is now to be solved with the boundary conditions at $z = 0$, $t > 0$

$$P = a e^{i\omega t} + b e^{-i\omega t} \quad \ldots(22)$$

$$Q = c e^{i\omega t} + d e^{-i\omega t} \quad \ldots(23)$$

and as $z \to \infty$, $t > 0$

$$P = 0, \quad Q = 0 \quad \ldots(24)$$

and with the initial condition at

$$t = 0 \text{ and all } z \geqslant 0,\ P = Q = \frac{\partial P}{\partial t} = \frac{\partial Q}{\partial t} = 0. \quad \ldots(25)$$

Solution

In view of the impulsive nature of the problem, the method of Laplace transform is employed.

The Laplace transforms $\bar{P}(z, s)$ and $\bar{Q}(z, s)$ of $P(z, t)$ and $Q(z, t)$ satisfy

$$\left\{\eta\nu \frac{\partial^4}{\partial z^4} - [(\eta + \nu) s + 2\Omega i\eta + nB_0] \frac{\partial^2}{\partial z^2} + s(s + 2\Omega i)\right\} \begin{bmatrix} \bar{P}(z, s) \\ \bar{Q}(z, s) \end{bmatrix} = 0. \quad \ldots(26)$$

The solution for $\bar{P}(z, s)$ and $\bar{Q}(z, s)$ satisfying the given boundary and initial conditions are obtained as follows:

$$\bar{P}(z, s) = \left(\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right) \frac{nB_0\nu}{(\nu - \eta)(2\Omega i\eta + nB_0)} e^{-r_1 z}$$

$$+ \left(\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right) \left\{\frac{(\nu - \eta)\, 2\Omega i\eta - \eta nB_0}{(\nu - \eta)(2\Omega i\eta + nB_0}\right\} e^{-r_2 z} \quad \ldots(27)$$

$$\bar{Q}(z, s) = \left(\frac{c}{s - i\omega} + \frac{d}{s + i\omega}\right) \frac{nB_0\nu}{(\nu - \eta)(2\Omega i\eta + nB_0)} e^{-r_1 z}$$

$$+ \left(\frac{c}{s - i\omega} + \frac{d}{s + i\omega}\right) \left\{\frac{(\nu - \eta)\, 2\Omega i\eta - \eta nB_0}{(\nu - \eta)(2\Omega i\eta + nB_0)}\right\} e^{-r_2 z} \quad \ldots(28)$$

where r_1 and r_2 are given by

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \frac{1}{2\nu\eta} \{(\eta + \nu)\, s + 2\Omega i\eta + nB_0 \pm [\{(\eta + \nu)\, s$$

$$+ 2\Omega i\eta + nB_0\}^2 - 4\eta\nu s(s + 2\Omega i)]^{1/2}\}. \quad \ldots(29)$$

For the evaluation of the inversion integrals the nature of the boundary layers due to flow field $P(z, t)$ and magnetic field $Q(z, t)$ are obtained for small and large times by the behaviour of its Laplace transform $\bar{P}(z, s)$, $\bar{Q}(z, s)$ for large and small value of $|s|$ respectively.

(a) *For Small Times*

$$\bar{P}(z, s) \sim \left(\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right) \frac{nB_0\nu}{(\nu - \eta)(2\Omega i\eta + nB_0)} \exp(-s^{1/2}\nu^{-1/2}z)$$

$$+ \left(\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right) \left\{\frac{(\nu - \eta)\, 2\Omega i\eta - \eta nB_0}{(\nu - \eta)(2\Omega i\eta + nB_0)}\right\} \exp(-s^{1/2}\eta^{-1/2}z) \quad \ldots(30)$$

$$\bar{Q}(z, s) \sim \left(\frac{c}{s - i\omega} + \frac{d}{s + i\omega}\right) \frac{nB_0\nu}{(\nu - \eta)(2\Omega i\eta + nB_0)} \exp(-s^{1/2}\nu^{-1/2}z)$$

$$+ \left(\frac{c}{s - i\omega} + \frac{d}{s + i\omega}\right) \left\{\frac{(\nu - \eta)\, 2\Omega i\eta - \eta nB_0}{(\nu - \eta)(2\Omega i\eta - nB_0)}\right\} \exp(-s^{1/2}\eta^{-1/2}z) \quad \ldots(31)$$

with the aid of the table of inverse Laplace transforms (Erdelyi *et al.* 1953, vol. 1) the velocity distribution for small times is obtained as

$$P(z, t) \sim \frac{nB_0\nu}{(\nu - \eta)(2\Omega i\eta + nB_0)} \{\tfrac{1}{2}\, a\, e^{i\omega t} [\exp(z\nu^{-1/2}(i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2}\nu^{-1/2}t^{-1/2}$$

$$+ (i\omega t)^{1/2}) + \exp(-z\nu^{-1/2}(i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2}\, z\nu^{-1/2}t^{-1/2} - (i\omega t)^{1/2})]$$

$$+ \tfrac{1}{2}\, b\, e^{i\omega t} [\exp(z\nu^{-1/2}(-i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2}\, z\nu^{-1/2}t^{-1/2} + (-i\omega t)^{1/2}) +$$

(equation continued on p. 424)

$$+ \exp(-z\nu^{-1/2}(-i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2} z\nu^{-1/2}t^{-1/2} - (-i\omega t)^{1/2})]\}$$

$$+ \frac{(\nu - \eta)\, 2\Omega i\eta - \eta n B_0}{(\nu - \eta)(2\Omega i\eta + nB_0)} \{\tfrac{1}{2} a\, e^{i\omega t} [\exp(z\eta^{-1/2}(i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2}\eta^{-1/2}t^{-1/2}$$

$$+ (i\omega t)^{1/2}) + \exp(-z\eta^{-1/2}(i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2} z\eta^{-1/2}t^{-1/2} - (i\omega t)^{1/2})$$

$$+ \tfrac{1}{2} b\, e^{i\omega t} [\exp(z\eta^{-1/2}(-i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2} z\eta^{-1/2}t^{-1/2} + (-i\omega t)^{1/2})$$

$$+ \exp(-z\eta^{-1/2}(-i\omega)^{1/2}) \operatorname{Erfc}(\tfrac{1}{2} z\eta^{-1/2}t^{-1/2} - (-i\omega t)^{1/2})]\}. \quad \ldots(32)$$

Similarly, solution for $Q(z, t)$ can also be obtained.

Is is evident from the result (32) that immediately after the impulsive motion is imposed on the disk, an unsteady boundary layer flow is generated in the vicinity of the disk. And solution consists of two distinct boundary layers of thickness of the order $\left(\frac{\nu}{\omega}\right)^{1/2}$ and $\left(\frac{\eta}{\omega}\right)^{1/2}$. The first is classical Stokes layer and second is a new boundary layer [first predicted by Sen and Debnath (1978)]. It is noted that these boundary layers are not at all affected by the external magnetic field or rotation.

(b) *For Large Times*

$$\bar{P}(z, s) \sim a\, e^{-\lambda z} \frac{1}{s + i\omega} e^{-smz} + b\, e^{-\lambda z} \frac{1}{s + i\omega} e^{-smz} \quad \ldots(33)$$

where

$$\lambda = \left(\frac{2\Omega i\eta + nB_0}{\nu\eta}\right)^{1/2}$$

$$m = \frac{2\eta^2\Omega i + (\eta + \nu)\, nB_0}{2(2\Omega i\eta + nB_0)^2}.$$

The Laplace-inversion of (33) is given by

$$P(z, t) \sim 0 \quad \text{for} \quad t < mz,$$

$$\sim a \exp(-(\lambda + i\omega m)\, z + i\omega t)) + b \exp(-(\lambda - i\omega m)\, z - i\omega t)) \quad \text{for} \quad t > mz. \quad \ldots(34)$$

Now

$$\lambda \pm i\omega m = \left(\frac{\Omega}{\nu}\right)^{1/2} \left[\left\{\left(\frac{n^2B_0^2}{4\Omega^2\eta^2} + 1\right)^{1/2} + \frac{nB_0}{2\Omega\eta}\right\}^{1/2} + i\left\{\left(\frac{n^2B_0^2}{4\Omega^2\eta^2} + 1\right)^{1/2} - \frac{nB_0}{2\Omega\eta}\right\}^{1/2}\right] \pm \left\{\frac{-\eta\Omega\omega nB_0(2\eta + \nu)}{(n^2B_0^2 + 4\Omega^2\eta^2)^2} + \frac{i[(\eta + \nu)(nB_0)^2 - 4\Omega^2\eta^3]}{2(n^2B_0^2 + 4\Omega^2\eta^2)^2}\right\}. \quad \ldots(35)$$

The real part α (say) is given by

$$\alpha = \left(\frac{\Omega}{\nu}\right)^{1/2} \left\{\left(\frac{n^2 B_0^2}{4\Omega^2\eta^2} + 1\right)^{1/2} + \frac{nB_0}{2\Omega\eta}\right\}^{1/2} \mp \frac{\eta\Omega\omega n B_0(2\eta + \nu)}{(n^2 B_0^2 + 4\Omega^2\eta^2)^2}. \quad \ldots(36)$$

If α is positive quantity than $(1/\alpha)$ is, in this case, hydromagnetic boundary layer thickness.

This solution clearly suggests that a steady state is attained in the limit $t \to \infty$ with $\Omega \neq 0$, $\omega = 0$.

Thus

$$P(z, t) = Ue^{-\lambda z}, \quad \ldots(37)$$

where $a + b = U$ is a complex constant.

The associated hydromagnetic Ekman layer has a thickness of penetration of vorticity of order

$$\left(\frac{\nu}{\Omega}\right)^{1/2} \left[\left(\frac{nB_0}{4\Omega^2\eta^2} + 1\right)^{1/2} - \frac{nB_0}{2\Omega\eta}\right]^{1/2}.$$

which is less than the classical Ekman layer. This confirms that the rotation and the magnetic field have pronounced effects on the formation of boundary layers.

Finally, solution (37) shows an exact agreement with that of the hydrodynamic problem.

Similar results are valid for $Q(z, t)$, in agreement with those of Debnath (1972) and Sen and Debnath (1978).

Acknowledgement

Authors are thankful to Dr R. S. Pathak for his help and valuable suggestions.

References

Debnath, L. (1972). On unsteady magnetohydrodynamic boundary layers in a rotating flow. *ZAMM*, **52**, 623.

——— (1973). On Ekman and Hartman boundary layers in a rotating fluid. *Acta Mech.*, **18**, 333–41.

——— (1975a). Exact solutions of the unsteady hydrodynamic and hydromagnetic boundary layer equations in a rotating fluid system. *ZAMM*, **55**, 431-38.

——— (1975b). Intertial oscillations and hydromagnetic multiple boundary layers in a rotating fluid. *ZAMM*, **55**, 141–47.

Hide, R., and Roberts, P. H. (1960). Hydromagnetic flow due to an oscillating plane. *Rev. Mod. Phys.*, **32**, 799.

Sen, S., and Debnath, L. (1978). On unsteady magnetic hydrodynamic boundary layer flows. *Bull. Calcutta math. Soc.*, **70**, 11–22.

Thornely, C. (1968). On Stokes and Rayleigh layers in a rotating system. *Q. Jl Mech. appl. Math.*, **21**, 451.