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AUTO-PARALLELISM AND UNION GAUSSIAN CURVATURE IN RIEMANNIAN SPACE

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A generalization of Riemann tensor called the union Gaussian curvature is introduced and it is shown that for a vector of a two dimensional space in a Riemann manifold to return to its original position, when transported by union parallelism, the manifold need not be union flat.

INTRODUCTION

In a Riemannian manifold it is very well known that if a vector of a two dimensional space V_2 is transported by Levi-Civita parallelism around an infinitesimal circuit then it returns to its original position if and only if the components of the Riemann tensor vanish i.e. the manifold is flat.

In the present paper we shall introduce a generalization of the Riemann tensor called the union Gaussian curvature and it will be shown that if the vector returns to its original position when transported by union parallelism then the manifold need not be union flat.

1. UNION PARALLEL DISPLACEMENT

Let a two dimensional Riemannian surface V_2 given by the equations $x^i = x^i(u^\sigma)$, $(i = 1, 2, 3;\ \sigma = 1, 2)$ be immersed in a V_3. Suppose λ^i is a vector field defining a congruence of curves in V_3. At the point of the space V_3 we may write,

$$\lambda^i = p^\alpha X^i_{,\alpha} + qN^i$$

where $x^i_{,\alpha} = \dfrac{\partial x^i}{\partial u^\alpha}$ an N^i is the normal to the surface. Springer (1956) has obtained the differential equation of the union curve relative to λ^i in the form,

$$\frac{d^2u^\gamma}{ds^2} + U^\gamma_{\beta\tau}\,\frac{du^\beta}{ds}\cdot\frac{du^\tau}{ds} = 0 \qquad \text{...(1.1)}$$

where

$$U^\gamma_{\beta\tau} = \Gamma^\gamma_{\beta\tau} + K_n b^\sigma g_{\alpha\beta}\epsilon_{\sigma\tau}\epsilon^{\alpha\gamma} \qquad \text{...(1.2)}$$

$$\epsilon_{\sigma\tau} = 0 \text{ if } \sigma = \tau \text{ and } \epsilon_{12} = 1 = -\epsilon_{21}$$

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K_n is the normal curvature of the surface and $b^\sigma = p^\sigma/q$. The connection parameter known as the union connection parameter may be used in defining the following two methods of covariant differentiations of a tensor of the type $T^\alpha_\beta(u)$

$$T^\alpha_{\beta;\gamma} = \frac{\partial T^\alpha_\beta}{\partial u^\gamma} + T^\rho_\beta U^\alpha_{\rho\gamma} - T^\alpha_\rho U^\rho_{\beta\gamma} \qquad ...(1.3)$$

and

$$T^\alpha_{\beta:\gamma} = \frac{\partial T^\alpha_\beta}{\partial u^\gamma} + T^\rho_\beta U^\alpha_{\gamma\rho} - T^\alpha_\rho U^\rho_{\gamma\beta} \,. \qquad ...(1.4)$$

A vector field v^α is said to undergo the U-parallelism of the first kind if

$$v^\alpha_{;\beta} \frac{du^\beta}{ds} = 0.$$

It undergoes U-parallelism of the second kind if,

$$v^\alpha_{:\beta} \frac{du^\beta}{ds} = 0.$$

It follows from eqn. (1.1) that a union curve is autoparallel relative to each of the two methods of covariant differentiation.

2. Union Gaussian Curvature

Springer (1956, 1945) has studied union curves and union parallelism of a vector field and union Gaussian curvature with the help of symmetric part of the union connection parameter $U^\alpha_{\beta\gamma}$. On the other hand Gorowara (1968) has used the asymmetric connection parameter $U^\alpha_{\beta\gamma}$ and the process of covariant differentiation given by eqn. (1.3) for studying similar properties of the space V_2.

In the present paper we consider the second kind of covariant derivative given by the eqn. (1.4) for obtaining union Gaussian curvature and study some of its properties.

If the vector λ^σ be transported by Levi-Civita parallelism around an infinitesimal, parallelogram determined by du^σ and $'du^\sigma$ then the change $\Delta\lambda^\sigma$ at $P(u^\sigma)$ is given by

$$\Delta\lambda^\sigma = {}'d(d\lambda^\sigma) - d('d\lambda^\sigma) = R^\sigma_{\tau\alpha\beta}\, {}'du^\alpha du^\beta \lambda^\tau$$

up to infinitesimals of higher order, where $R^\sigma_{\cdot\tau\alpha\beta}$ are the components of the Riemann tensor.

Proceeding as in Eisenhart (1949) we define the union Gaussian curvature by

$$\bar{K} = \tfrac{1}{4}\, \bar{R}_{\rho\tau\alpha\beta}\, \epsilon^{\rho\tau}\, \epsilon^{\alpha\beta} \qquad \text{...(2.1)}$$

where

$$\bar{R}^{\sigma}_{\cdot\tau\alpha\beta} = \frac{\partial U^{\sigma}_{\alpha\tau}}{\partial u^{\beta}} - \frac{\partial U^{\sigma}_{\beta\tau}}{\partial u^{\alpha}} - U^{\sigma}_{\alpha\gamma}\, U^{\gamma}_{\beta\tau} + U^{\sigma}_{\beta\gamma}\, U^{\gamma}_{\alpha\tau} \qquad \text{...(2.2)}$$

is the union curvature tensor of the subspace. After putting $K_n b^{\alpha} = L^{\alpha}$ eqn. (1.2) takes the form,

$$U^{\sigma}_{\tau\beta} = \Gamma^{\sigma}_{\tau\beta} + L^{\nu} g_{\delta\tau}\, \epsilon_{\nu\beta}\, \epsilon^{\delta\sigma}. \qquad \text{...(2.3)}$$

Simplifying (2.1) with the help of eqns. (2.2), (2.3) and the relations

$$\epsilon^{\gamma\alpha} g_{\gamma\beta} = \epsilon_{\beta\sigma}\, \epsilon^{\alpha\sigma} \quad \text{and} \quad \epsilon^{\alpha\gamma}\, \epsilon_{\beta\gamma} = \delta^{\alpha}_{\beta}$$

we get,

$$2\bar{K} = 2K + \frac{\partial L^{\nu}}{\partial u^{\nu}} - L_{\sigma}(L_{\beta\alpha\delta}\, \epsilon^{\sigma\delta}\, \epsilon^{\alpha\beta} + \Gamma^{\sigma}_{\beta\gamma}\, g^{\gamma\beta}). \qquad \text{...(2.4)}$$

The vector λ^{ν}, after being transported by union parallel displacement returns to its original position if all the components $\bar{R}^{\sigma}_{\cdot\tau\alpha\beta}$ are zero or equivalently $\bar{K} = 0$ in which case eqn. (2.4) reduces to

$$2K + \frac{\partial L^{\nu}}{\partial u^{\nu}} - L_{\sigma}(L_{\beta\alpha\delta}\, \epsilon^{\sigma\delta}\, \epsilon^{\alpha\beta} + \Gamma^{\sigma}_{\beta\gamma}\, g^{\gamma\beta}) = 0. \qquad \text{...(2.5)}$$

Since eqn. (2.5) imposes only one condition on L^1 and L^2 out of which one may be chosen arbitrarily, it follows that the manifold need not be union flat.

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References

Eisenhart, L. P. (1949). Riemannian Geometry. Princeton University Press, Princeton, p. 67.

Gorowara, K. (1968). Union Gaussian curvature. *Tensor, N.S.*, **19**(2), 233–37.

Springer, C. E. (1945). Union curves and union curvature. *Bull. Am. math. Soc.*, **51**, 680–91.

———— (1956). Union parallel displacement. Recibido el 15–VIII, University of Oklahoma.