Indian J. pure appl. Math., **12**(4) : 440–447, April 1981

FIXED POINT THEOREMS FOR MAPPINGS IN A UNIFORM SPACE WITH A CONTRACTIVE ITERATE

CHUMKI PANJA AND A. P. BAISNAB
Department of Mathematics, University of Burdwan, Burdwan

(*Received* 7 *May* 1980)

In this paper fixed point theorems have been proved for a mapping $T : X \to X$ by taking (i) some iterate of T satisfying a contractive type condition, and (ii) X to be a Hausdorff uniform space. The metric analogues of our results then include those already known.

1. INTRODUCTION

Let (x, d) be a metric space. Let $\epsilon > 0$ and $0 < \lambda < 1$. Then according to Edelstein (1961) a mapping $T : X \to X$ is called (ϵ, λ) — uniformly locally contractive if $d(T(x), T(y)) \leqslant \lambda d(x, y)$ whenever $d(x, y) < \epsilon$. He has proved a fixed point theorem for such a mapping by taking (X, d) to be a ϵ-chainable complete metric space. In a multi-valued setting Covitz and Nadler (1970) have extended this result by taking (X, d) to be generalized metric space and obtained the following theorem :

Theorem 1.1 — Let (X, d) be a generalized complete metric space and let $x_0 \in X$. If $F : X \to cL(X)$ is a uniformly locally contractive multi-valued mapping, then the following alternative holds :

Either (i) for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_0,

$$d(x_{i-1}, x_i) \geqslant \epsilon \text{ for each } i = 1, 2, \ldots$$

or (ii) there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of F at x_0 such that $\{x_i\}$ converges to a fixed point of F.

It is also known that a fixed point theorem for a mapping $T : X \to X$ can be proved by assuming some iterate of T to satisfy a contractive type condition. Our purpose of this paper is to establish fixed point theorems for a mapping $T : X \to X$ by assuming some iterate of T to satisfy a contractive type condition, and by taking X to be a Hausdorff uniform space. Our Theorem 3.2 and Theorem 3.4 can therefore be looked upon as extentions of Theorem of Sehgal (1969) in the same way as Edelstein's fixed point theorems on contractive mappings as extentions of contraction principle theorem. Then metric analogues of our theorems include the results of Edelstein (1962). A corollary resulting from one of the metric analogues thereby reveals that fixed point theorem as proved by Roy and Rhoades (1977) does no longer stand.

Before proving our results we give some preliminary definitions and results.

2. Preliminary Definitions and Results

Let $(X, \mathcal{U})$ be a Hausdorff uniform space. $(X, \mathcal{U})$ is called sequentially complete if every Cauchy sequence in X converges to a point in X.

For any pseudometric p on X and any $r > 0$, we write

$$V_{(p,r)} = \{(x, y);\ x, y \in X \text{ and } p(x, y) < r\}$$

From Theorem 15 of Kelly (1955, p. 188) we see that the uniformity $\mathcal{U}$ on X can be generated by the family $\mathcal{P}$ of all pseudometrices on X which are uniformly continuous on $X \times X$.

Let $\mathcal{P}$ be a family of pseudometrices on X generating the uniformity $\mathcal{U}$. Denote by $\mathcal{V}$ the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i, r_i)}$, where $p_i \in \mathcal{P}$ and

$$r_i > 0,\ i = 1, 2, \ldots, n$$

(the integer n is not fixed). Then clearly $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$.

Let $V \in \mathcal{V}$. Then $V = \bigcap_{i=1}^{n} V_{(p_i, r_i)}$ where $p_i \in \mathcal{P}$ and $r_i > 0$, $i = 1, 2, \ldots n$.

For each $\alpha > 0$, the set $\bigcap_{i=1}^{n} V_{(p, \alpha r_i)}$ belongs to $\mathcal{V}$. We denote this set by αV.

Acharya (1974) has proved the following Lemmas which have been used in the proofs of our Theorems to follow.

Lemma 2.1 — If $V \in \mathcal{V}$ and α, β are positive, then $\alpha(\beta V) = (\alpha\beta) V$.

Lemma 2.2 — If $V \in \mathcal{V}$ and α, β are positive and $\alpha < \beta$, then $\alpha V \subset \beta V$.

Lemma 2.3 — Let p be any pseudometric on X and α, β be any two positive numbers. If $(x, y) \in \alpha V_{(p, r_1)} \circ \beta V_{(p, r_2)}$ then $p(x, y) < \alpha r_1 + \beta r_2$.

Lemma 2.4 — If $V \in \mathcal{V}$ and α, β are positive, then $\alpha V \circ \beta V \subset (\alpha + \beta) V$.

Lemma 2.5 — Let $(x, y) \in X \times X$. Then for any V in $\mathcal{V}$ there is a positive number λ such that $(x, y) \in \lambda V$.

Lemma 2.6 — Let V be any member of $\mathcal{V}$. Then there is a pseudometric p on X where $p(x, y) = \operatorname{Inf}\{\lambda > 0;\ (x, y) \in \lambda V\}$ for $(x, y) \in X \times X$.

We call the pseudometric p of Lemma 2.6 the Minkowski's pseudometric of V.

3. Results on Fixed Point

Theorem 3.1 — Let $(X, \mathcal{U})$ be a Hausdorff Uniform space which is sequentially complete. Let W be a member of $\mathcal{V}$, and $0 < \alpha < 1$ be given. Let T_1, T_2 be two mappings of X into itself such that for any $x, y \in X$, there exist positive integers $n(x)$ and $m(y)$ with the property that for $V(\subseteq W)$ in $\mathcal{V}$ and

$$(x, y) \in W, (T_1^{m(x)}(x), T_2^{n(y)}(y)) \in \alpha V.$$

Then the following alternative holds :

Either (i) $(x_n, x_{n+1}) \notin W$ for each n, or (ii) the sequence $\{x_n\}$ converges to a common periodic point of T_1 and T_2, where $\{x_n\}$ is defined as $x_{2n+1} = T_1^{m(x_{2n})}(x_{2n})$ and

$$x_{2n+2} = T_2^{n(x_{2n+1})}(x_{2n+1}),\ n = 0, 1, 2, \ldots \text{ and } x_0 \in X.$$

Proof : Define $F_1, F_2 : X \to X$ as follows :

$$F_1(x) = T_1^{m(x)}(x) \text{ and } F_2(x) = T_2^{n(x)}(x).$$

Let $x_0 \in X$, then the sequence $\{x_n\}$ is given by $x_{2n+1} = F_1(x_{2n})$ and $x_{2n+2} = F_2(x_{2n+1})$, $n = 0, 1, 2, \ldots$ choose V in $\mathcal{V}$ such that $V \subseteq W$. Now suppose that (i) does not hold. Then $(x_N, x_{N-1}) \in W$ for some integer $N \geqslant 1$. Let $N = 2p$. Then $(x_{2p}, x_{2p-1}) \in W$ implies that $(F_1(x_{2p}), F_2(x_{2p-1})) = (x_{2p+1}, x_{2p}) \in \alpha V \subset V \subseteq W$. This also gives $(F_2(x_{2p+1}), F_1(x_{2p})) = (x_{2p+2}, x_{2p+1}) \in \alpha(\alpha V) = \alpha^2 V \subset V \subseteq W$. Similarly if $N = 2p + 1$ then we have also $(x_{2p+1}, x_{2p+2}) \in \alpha^2 V$ i.e. $(x_{N+1}, x_{N+2}) \in \alpha^2 V$. Continuing the process we produce a sequence $\{x_{N+i}\}_{i=1}^{\infty}$ of points of X such that $(x_{N+i}, x_{N+i+1}) \in \alpha^{i+1} V$ for all i. Since $0 < \alpha < 1$, we choose a positive integer i_0 such that $\alpha^{i+1} < 1$ for $i \geqslant i_0$. Then $(x_{N+i}, x_{N+i+1}) \in V \subset W$ for $i \geqslant i_0$. Thus $\{x_n\}$ is cauchy. As X is sequentially complete, $\lim_{n\to\infty} x_n = \xi \in X$. As $\lim_{i\to\infty} x_{2i} = \xi$, there exists an integer i' such that $(x_{2i}, \xi) \in W$ for $i \geqslant i'$. This implies that $(F_1(x_{2i}), F_2(\xi)) = (x_{2i+1}, F_2(\xi)) \in \alpha V \subset V$ for $i \geqslant i'$. So, $\lim_{i\to\infty} x_{2i+1} = F_2(\xi)$. Therefore, $\xi = F_2(\xi)$. Similarly, $\xi = F_1(\xi)$. Thus $\xi = T_1^{m(\xi)}(\xi) = T_2^{n(\xi)}(\xi)$ is a common periodic point of T_1 and T_2, and $\lim_{n\to\infty} x_n = \xi$.

Corollary 3.1 — Let (X, d) be a complete metric space and T_1, T_2 be two mappings of X into itself such that for $x, y \in X$, there exist positive integers $m(x)$ and $n(y)$ such that

$$d(T_1^{m(x)}(x), T_2^{n(y)}(y)) \leqslant \alpha d(x, y),\ 0 < \alpha < 1.$$

Then T_1, T_2 have a common periodic point ξ in X, and for any $x_0 \in X$, $\lim_{n\to\infty} x_n = \xi$, where

$$x_{2n+1} = T_1^{m(x_{2n})}(x_{2n}) \text{ and } x_{2n+2} = T_2^{n(x_{2n+1})}(x_{2n+1}),\ n = 0, 1, 2, \ldots$$

The conclusion of corollary 3.1 is the best possible in the sense that common periodic point of T_1 and T_2 is not necessarily a common fixed point of T_1 and T_2. This observation has also been made by S. Kasahara leading to an errata [*Pacific J. Math.*, 79, No. 2 (1978), p. 563] in respect of Theorem 1 (involving mapping of the type as in Corollary 3.1 above) obtained by Roy and Rhoades (1977). In consequence of this remark it also follows that contrary to the claim of Roy and Rhoades, Sehgal's theorem is not included in Theorem 1 of Roy and Rhoades (1977). The following example illustrates our contention.

Example 3.1 — Let $X = \{1, 2, 3, 4\}$ and (X, d) be a metric space where

$$d(1, 2) = \tfrac{3}{7},\ d(1, 3) = \tfrac{5}{7},\ d(1, 4) = \tfrac{5}{7},\ d(2, 3) = \tfrac{2}{7},\ d(2, 4) = \tfrac{4}{7},$$

$$d(3, 4) = \tfrac{2}{7},\ d(1, 1) = d(2, 2) = d(3, 3) = d(4, 4) = 0.$$

Let $T : X \to X$ be defined as follows :

$$T(1) = 2,\ T(2) = 3,\ T(3) = 4 \text{ and } T(4) = 4.$$

Let $\quad n(1) = 1,\ n(2) = 2,\ n(3) = 3 \text{ and } n(4) = 4.$

Here taking $\alpha = 6/7$, T satisfies all conditions of Theorem of Sehgal (1969) but T does not fit in corollary 3.1 above where in we assume $T_1 = T_2$ and $m = n$, because T fails to satisfy the inequality for $x = 1$ and $y = 2$.

We conclude this section by proving the following fixed point theorems which show the change in type incorporated into the mappings as appeared in corollary 3.1.

Theorem 3.2 — Let $(X, \mathcal{U})$ be a Hausdorff uniform space. Suppose $T : X \to X$ is continuous, $n : X \to N^+$ (set of all positive integers) is continuous and for any V in $\mathcal{V}$ with $(x, y) \in V$ there exists $\lambda(x, y)$ with $0 < \lambda(x, y) < 1$ such that $(T^{n(x)}(x), T^{n(x)}(y)) \in \lambda(x, y)\, V$. If there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n;\ x_{n+1} = T^{n(x_n)}(x_n),\ x_0 \in X\}$, converging to an element $z_0 \in X$, then z_0 is the unique fixed point of T, and $\lim_{n\to\infty} x_n = z_0$.

PROOF : Let V be any member of $\mathcal{V}$. Denote by p the Minkowski's pseudometric of V. Let $x, y \in X$. Write $p(x, y) = r$ and take $\epsilon > 0$. Then $(x, y) \in (r + \epsilon)\, V)$. So by the given condition $(T^{n(x)}(x), T^{n(x)}(y)) \in \lambda(x, y)\, \{(r + \epsilon)\, V\}$. Hence we have $p(T^{n(x)}(x), T^{n(x)}(y)) < \lambda(x, y)\, \{r + \epsilon\}$. Since $\epsilon > 0$ is arbitrary, $p(T^{n(x)}(x), T^{n(x)}(y)) \leqslant \lambda(x, y).\ r = \lambda(x, y)\, p(x, y)$. Hence $p(T^{n(x)}(x), T^{n(x)}(y)) < p(x, y)$. Next choose $U \in \mathcal{V}$ so that $U \circ U \circ U \subset V$. Now $\lim_{i \to \infty} x_{n_i} = z_0$ implies that there is an index i_0 such that $(x_{n_i}, z_0) \in U$ for $i \geqslant i_0$. So we have

$$(T^{n(x_{n_i})}(x_{n_i}), T^{n(x_{n_i})}(z_0)) \in \lambda(x_{n_i}, z_0)\, U \subset U.$$

for $i \geqslant i_0$. Again continuity of $n : X \to N^+$ implies that for sufficiently large values of i, $(T^{n(x_{n_i})}(z_0), T^{n(z_0)}(z_0)) \in U$. Thus for $i \geqslant i_0$ we have $(T^{n(x_{n_i})}(x_{n_i}), T^{n(z_0)}(z_0)) = (x_{n_i+1}, z_1) \in U \circ U \subset V$. Hence $\lim_{i \to \infty} x_{n_i+1} = z_1$. Again for $(x_0, T(x_0)) \in X \times X$ there is a positive number μ such that $(x_0, T(x_0)) \in \mu V$. Put $\mu V = V'$. Then $V' \in \mathcal{V}$. We have $p(x_1, T(x_1)) = p(T^{n(x_0)}(x_0), T^{n(x_0)}(T(x_0))) < p(x_0, T(x_0))$ and $p(x_2, T(x_2)) = p(T^{n(x_1)}(x_1), T^{n(x_1)}(T(x_1))) < p(x_1, T(x_1))$. Thus

$$p(x_2, T(x_2)) < p(x_1, T(x_1)) < p(x_0, T(x_0)).$$

continuing in this way we obtain, $p(x_n, T(x_n)) < p(x_{n-1}, T(x_{n-1})) < \ldots < p(x_1, T(x_1)) < p(x_0, T(x_0))$. Thus $\{p(x_n, T(x_n)\}$ is a decreasing sequence of nonnegative terms and therefore convergent. Let $\lim_{n \to \infty} p(x_n, T(x_n)) = d$. If $d = 0$, then $p(z_0, T(z_0)) = 0$. So $(z_0, T(z_0)) \in V$. Since V is arbitrary and X is Hausdorff, we have $z_0 = T(z_0)$. Next, if possible, let $d > 0$. Then proceeding as before, we have

$$p(z_{n+1}, T(z_{n+1})) < p(z_n, T(z_n)) < \ldots < p(z_1, T(z_1)) < p(z_0, T(z_0)).$$

Now $p(z_0, T(z_0)) = p(\lim_{i\to\infty} x_{n_i}, \lim_{i\to\infty} T(x_{n_i})) = \lim_{i\to\infty} p(x_{n_i}, T(x_{n_i})) = d = \lim_{i\to\infty} p(x_{n_i+1}, T(x_{n_i+1})) = p(z_1, T(z_1))$ which contradicts that $p(z_1, T(z_1)) < p(z_0, T(z_0))$. Hence $d = 0$. Thus $z_0 = T(z_0)$. It can be easily seen that z_0 is the unique fixed point of T.

Next to establish $\lim_{j\to\infty} x_j = z_0$, take any $V \in \mathcal{V}$. $\lim_{i\to\infty} x_{n_i} = z_0 = T(z_0)$ implies there is an index N such that $(x_{n_i}, z_0) \in V$ for $i \geqslant N$. If $j > n_N$ put $j = n_N + l$. Now $(x_{n_N}, z_0) \in V$ implies that $(T^{n(x_{n_N})}(x_{n_N}), T^{n(x_{n_N})}(z_0)) = (x_{n_N+1}, z_0) \in \lambda(x_{n_N}, z_0)\, V \subset V$. Again, $(x_{n_N+1}, z_0) \in V$ implies that $(x_{n_N+2}, z_0) \in \lambda(x_{n_N+1}, z_0)\, V \subset V$. Proceeding in this way we obtain, $(x_{n_N+l}, z_0) \in V$ i.e., $(x_j, z_0) \in V$. Thus $\lim_{j\to\infty} x_j = z_0$. This completes the proof.

Metric analogue of Theorem 3.2 reads as :

Theorem 3.3 — Let (X, d) be a metric space and T be a continuous self mapping of X such that there is a continuous function $n : X \to N^+$ satisfying $d(T^{n(x)}(x), T^{n(x)}(y)) < d(x, y)$ for $x, y \in X$. If there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$

where $x_{n+1} = T^{n(x_n)}(x_n)$, converging to a point $z_0 \in X$, then z_0 is the unique fixed point of T and $\lim_{n\to\infty} x_n = z_0$.

Theorem 3.3 generalizes Theorem 1 of Edelstein (1962).

Theorem 3.4 — Let $(X, \mathcal{U})$ be a Hausdorff Uniform space. Suppose $T : X \to X$ is continuous, and $n : X \to N^+$ is continuous. If $W \in \mathcal{V}$ and $(x, y) \in W$ then for every $V \in \mathcal{V}$ with $V \subset W$ and $(x, y) \in V$, suppose there exists $\lambda(x, y)$ with $0 < \lambda(x, y) < 1$ satisfying $(T^{n(x)}(x), T^{n(x)}(y)) \in \lambda(x, y) V$. Suppose further that $\{x_n = T^{n(x_{n-1})}(x_{n-1})\}$ with $x_0 \in X$ has a subsequence $\{x_{n_i}\}$ converging to $\xi \in X$, then ξ is a periodic point of T.

PROOF : Take any $V \in \mathcal{V}$ with $V \subset W$ and choose $U \in \mathcal{V}$ so that

$$U \circ U \circ U \circ U \subset V.$$

Now $\lim_{i\to\infty} x_{n_i} = \xi$ implies that there is an index N such that $(x_{n_i}, \xi) \in U$ for $i \geqslant N$. Choose any $i \geqslant N$. Then by the given condition,

$$(T^{n(x_{n_i})}(x_{n_i}), T^{n(x_{n_i})}(\xi)) = (x_{n_i+1}, T^{n(x_{n_i})}(\xi)) \in \lambda(x_{n_i}, \xi)\, U \subset U \subset W.$$

This again implies that $(x_{n_i+2}, T^{n(x_{n_i})+n(x_{n_i+1})}(\xi)) \in U \subset W$.

Proceeding in this way we have,

$$(x_{n_{i+1}}, T^{n(x_{n_i})+n(x_{n_i+1})+\cdots+n(x_{n_i+l-1})}(\xi)) \in U \subset W \text{ where } l = n_{i+1} - n_i.$$

Put $K = n(x_{n_i}) + n(x_{n_i+1}) + \ldots + n(x_{n_i+l-1})$. Then $(x_{n_{i+1}}, T^K(\xi)) \in U \subset W$. So, $(\xi, T^K(\xi)) \in U \circ U \subset V \subset W$. Next, if possible, let there exists a member $V' \in \mathcal{V}$ such that $(\xi, T^K(\xi)) \notin V'$. Let $p_{V'}$ denotes the Minikowski's pseudometric of V'. Let $(x, y) \in W$ and $p_{V'}(x, y) = r$. Take any $\epsilon > 0$. Then $(x, y) \in (r + \epsilon)\, V'$. Again $\{(r + \epsilon)\, V'\} \cap W \subset W$ and $(x, y) \in \{(r + \epsilon)\, V'\} \cap W$. So we have $(T^{n(x)}(x), T^{n(x)}(y)) \in \lambda(x, y)\, [\{(r + \epsilon)\, V'\} \cap W]$. Therefore

$$p_{V'}(T^{n(x)}(x), T^{n(x)}(y)) < \lambda(x, y)\, \{r + \epsilon\}.$$

Since $\epsilon > 0$ is arbitrary,

$$p_{V'}(T^{n(x)}(x), T^{n(x)}(y)) \leqslant \lambda(x, y)\, r < r = p_{V'}(x, y) \qquad \ldots(3.4.1)$$

Define a function $\gamma : X \times X - \Delta \to$ Reals by $\gamma(x, y) = \dfrac{p_{V'}(T^{n(x)}(x), T^{n(x)}(y))}{p_{V'}(x, y)}$. Then γ is continuous at $(\xi, T^K(\xi))$. Further $p_{V'}(\xi, T^K(\xi)) > 0$, because $(\xi, T^K(\xi) \notin V'$. Let $\gamma(\xi, T^K\xi)) < \delta < 1$. Further X being Hausdorff, using continuity of γ, we find $s_1 = s_1(\xi, \rho) = \{z \in X ; p_{V'}(\xi, z) < \rho\}$ and $s_2 = s_2(T^K(\xi), \rho) = \{z \in X ; p_{V'}(T^K(\xi), z) < \rho\}$ for a positive ρ with $s_1(\xi, \rho) \cap s_2(T^K(\xi), \rho) = \phi$ such that $\gamma(x_{n_j}, T^K(x_{n_j})) < \delta$ whenever $x_{n_j} \in s_1$ and $T^K(x_{n_j}) \in s_2$ for sufficiently large j. Since $\lim_{j\to\infty} x_{n_j} = \xi$ and $\lim_{j\to\infty} T^K(x_{n_j}) = T^K(\xi)$ we can find an index $\nu(\geqslant N)$ such that $x_{n_j} \in s_1$ and $T^K(x_{n_j}) \in s_2$ for $j \geqslant \nu$. Hence

$$\gamma(x_{n_j}, T^K(x_{n_j})) = \frac{p_{V'}(T^{n(x_{n_j})}(x_{n_j}), T^{n(x_{n_j})}(T^K(x_{n_j})))}{p_{V'}(x_{n_j}, T^K(x_{n_j}))} < \delta.$$

Therefore,

$$p_{V'}(x_{n_j+1}, T^K(x_{n_j+1})) < \delta p_{V'}(x_{n_j}, T^K(x_{n_j})) \qquad \ldots(3.4.2)$$

Again $\lim_{j\to\infty} T^K(x_{n_j}) = T^K(\xi)$. So there exists an index $N'(\geqslant \nu)$ such that $(T^K(x_{n_j}), T^K(\xi)) \in U$ for $j \geqslant N'$. Now $(x_{n_j}, \xi) \in U$ for $j \geqslant N$ together with

$$(\xi, T^K(\xi)) \in U \circ U$$

implies that for $j \geqslant N'(\geqslant \nu \geqslant N)$, $(x_{n_j}, T^K(x_{n_j})) \in U \circ U \circ U \circ U \subset V \subset W$. So we have $(T^{n(x_{n_j})}(x_{n_j}), T^{n(x_{n_j})}(T^K(x_{n_j})))$

$$= (x_{n_j+1}, T^K(x_{n_j+1})) \in \lambda(x_{n_j}, T^K(x_{n_j}))\, V \subset V \subset W$$

for $j \geqslant N'$ and from (3.4.1) we have $p_{V'}(x_{n_j+2}, T^K(x_{n_j+2})) < p_{V'}(x_{n_j+1}, T^K(x_{n_j+1}))$. Hence from (3.4.2) we obtain for $j \geqslant N'$,

$$p_{V'}(x_{n_j+2}, T^K(x_{n_j+2})) < p_{V'}(x_{n_j+1}, T^K(x_{n_j+1})) < \delta p_{V'}(x_{n_j}, T^K(x_{n_j})).$$

Choose any $j \geqslant N^1$ and let $m = n_{j+1} - n_j$. Then arguing as before, we obtain,

$$p_{V'}(x_{n_j+m}, T^K(x_{n_j+m})) < p_{V'}(x_{n_j+m-1}, T^K(x_{n_j+m-1})) < \ldots$$

$$\ldots < p_{V'}(x_{n_j+1}, T^K(x_{n_j+1})) < \delta p_{V'}(x_{n_j}, T^K(x_{n_j})).$$

So $p_{V'}(x_{n_{j+1}}, T^K(x_{n_j+1})) < \delta p_{V'}(x_{n_j}, T^K(x_{n_j}))$. Repeating this argument, we have for $s > j \geqslant N'$, $p_{V'}(x_{n_s}, T^K(x_{n_s})) < \delta^{s-j} p_{V'}(x_{n_j}, T^K(x_{n_j})) \to 0$ as $s \to \infty$. So, $p_{V'}(\xi, T^K(\xi)) \leqslant p_{V'}(\xi, x_{n_s}) + p_{V'}(x_{n_s}, T^K(x_{n_s})) + p_{V'}(T^K(x_{n_s}), T^K(\xi)) \to 0$ as $s \to \infty$. This shows that $(\xi, T^K(\xi)) \in V'$ which contradicts that $(\xi, T^K(\xi)) \notin V'$. So

$$(\xi, T^K(\xi)) \in V'.$$

Therefore $(\xi, T^K(\xi)) \in \bigcap_{V \in \mathcal{V}} V$ and since X is Hausdorff, $\xi = T^K(\xi)$. This completes the proof. Metric analogue of Theorem 3.4 reads as

Theorem 3.5 — Let (X, d) be a metric space and $\epsilon > 0$ be given. Suppose there is a continuous function $n : X \to N^+$ such that for each $x \in X$, $d(T^{n(x)}(x), T^{n(x)}(y)) < d(x, y)$ for all y with $d(x, y) < \epsilon$. If there is a point x_0 in X such that $\{x_n = T^{n(x_{n-1})}(x_{n-1})\}$ has a subsequence converging to an element ξ in X, then ξ is a periodic point of T.

Theorem 3.5 is a generalization of Theorem 2 of Edelstein (1962).

References

Acharya, S. P. (1974). Some results on fixed points in Uniform spaces. *Yokohama Math. J.*. **22**, 105–16.

Covitz, H., and Nadler, S. B. (Jr.) (1970). Multi-valued contraction mappings in generalised metric spaces. *Israel J. Math.* **8**, 5–11.

Edelstein, M. (1961). An extension of Banach's contraction principle. *Proc. Am. math. Soc.*, **12**, 7–10.

——— (1962). On fixed and periodic points under contractive mappings. *J. Lond. math. Soc.*, **37**, 74–79.

Kelly, J. L. (1955). General Topology, D. Van Nostrand Co., New York.

Roy, B. K., and Rhoades, B. E. (1977). Fixed point theorems for mappings with a contractive iterate. *Pacific J. Math.*, **71**, 517–20.

Sehgal, V. M. (1969). A fixed point theorem for mappings with a contractive iterate. *Proc. Am. math. Soc.*, **23**, 631–34.