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EXTENSION OF LEADER'S FIXED POINT PRINCIPLE TO UNIFORM SPACES

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The fixed point principle based on conditional uniform equivalence of orbits has been extended to uniform spaces. This is a direct generalization of a theorem due to Leader (1977).

In a recent paper Leader (1977) has established a fixed point principle using conditional uniform equivalence of orbits. In this note we extend this principle to uniform spaces and, in the process, we observe that a result of Knill (1965) in uniform space follows from this extension as a particular case.

Before we discuss the main results we recall a few definitions due to Leader (1977). We call (f, X, d) an operator if f is a self-map of the metric space (X, d). The graph of (f, X, d) is called complete if the Cauchyness of $\{x_n\}$ and $\{fx_n\}$ implies $x_n \to x$ and $fx_n \to fx$ for some $x \in X$. A fixed point is called contractive if it is the limit of every orbit in X.

The following fixed point principle has been established by Leader (1977).

Theorem 1 (Leader 1977) — Let (f, X, d) be an operator with complete graph and $c > 0$. If $d(f^n x, f^n y) \to 0$ uniformly for all $x, y \in X$ with $d(x, y) \leqslant c$, then the orbits $\{f^n x\}$ with $d(x, fx) \leqslant c$ are uniformly Cauchy and the orbit of x converges to a fixed point $p = fp$.

In the following we extend the above theorem to a uniform space and in doing so we use the notations due to Kelley (1955).

Theorem 2 — Let f be a self map of a uniform space (X, u). Suppose, for some fixed $U \in u$ and for any $V \in u$ such that $V \subset U$, there exists an integer $m = m(V)$ such that $f_2^n (U) \subset V$ for $n \geqslant m$.

If $(x, fx) \in U$, the sequence of iterates $\{f^n x\}$ is Cauchy and, further, if the graph of f is complete, then $\{f^n x\}$ converges to a fixed point of f.

Proof : Let $x_n = f^n x$. For $W \in u$ we can find a sequence $\{V_k\}$ from the members of u such that $V_1 \circ V_1 \subset U \cap W$ and $V_k \circ V_k \subset V_{k-1}$ for $k = 2, 3, \ldots,$

implying obviously that each $V_k \subset U$. Therefore, for each k there exists an integer n_k such that $f_2^{n_k}(U) \subset V_k$.

The sequence $\{n_k\}$ may be assumed to be increasing. Now, since $(x, x_1) \in U$, we have (x_n, x_{n+1}) and $(x_{n+1}, x_{n+2}) \in V_k$ for $n \geqslant n_k$. This implies that

$$(x_n, x_{n+2}) \in V_k^2 \subset V_{k-1}.$$

Since $n \geqslant n_k \geqslant n_{k-1}$, we have $(x_{n+2}, x_{n+3}) \in V_{k-1}$ and $(x_{n+2}, x_{n+4}) \in V_{k-1}$. Now $(x_n, x_{n+2}), (x_{n+2}, x_{n+3}) \in V_{k-1} \Rightarrow (x_n, x_{n+3}) \in V_{k-1}^2 \subset V_{k-2}$ and (x_n, x_{n+2}), $(x_{n+2}, x_{n+4}) \in V_{k-1}$ imply $(x_n, x_{n+4}) \in V_{k-1}^2 \subset V_{k-2}$. Moreover $(x_n, x_{n+1}), (x_n, x_{n+2}) \in V_{k-2}$ for $n \geqslant n_k \geqslant n_{k-1} \geqslant n_{k-2}$. Therefore $(x_n, x_{n+r}) \in V_{k-2}$ for $r \leqslant 2^2$, $n \geqslant n_k$. Since $n + r \geqslant n_k$, we have $(x_{n+r}, x_{n+r+4}) \in V_{k-2}$. Now $(x_n, x_{n+r}), (x_{n+r}, x_{n+r+4}) \in V_{k-2}$ imply $(x_n, x_{n+r+4}) \in V_{k-2}^2 \subset V_{k-3}$. Thus we write $(x_n, x_{n+s}) \in V_{k-3}$ where $s \leqslant 2^3$. Proceeding similarly we get $(x_n, x_{n+r}) \in V_1$, where $n \geqslant n_k$ and $r \leqslant 2^{k-1}$.

Choose an integer l such that $n_1 < 2^{l-1}$. Then, for $n \geqslant n_l$ and $r \leqslant n_1$, $(x_n, x_{n+r}) \in V_1$. Thus

$$(x_n, x_{n+n_1}) \in V_1,\ (x_{n+n_1}, x_{n+n_1+r}) \in V_1,\ r \leqslant n_1 \qquad \ldots(1)$$

implying

$$(x_n, x_{n+n_1+r}) \in V_1^2 \subset U \cap W \subset U \qquad \ldots(2)$$

whence by applying $f_2^{n_1}$ we obtain

$$(x_{n+n_1}, x_{n+2n_1+r}) \in f_2^{n_1}(U) \subset V_1. \qquad \ldots(3)$$

From (1) and (3) we derive

$$(x_n, x_{n+2n_1+r}) \in V_1^2 \subset U \cap W \qquad \ldots(4)$$

where $r \leqslant n_1$. We will now prove, by induction, that $(x_n, x_{n+mn_1+r}) \in U \cap W$ for all integers $m \geqslant 0$, $n \geqslant n_l$ and $r \leqslant n_1$. Assume now that for some integer m, $(x_n, x_{n+mn_1+r}) \in U \cap W \subset U$. Then applying $f_2^{n_1}$, we obtain

$$(x_{n+n_1}, x_{n+(m+1)n_1+r}) \in f_2^{n_1}(U) \subset V_1. \qquad \ldots(5)$$

From (1) and (5) we obtain

$$(x_n, x_{n+(m+1)n_1+r}) \in V_1^2 \subset U \cap W.$$

Thus, induction is complete and so we have $(x_n, x_{n+mn_1+r}) \in U \cap W$ for all integers $m \geqslant 0, n \geqslant n_1$ and $r \leqslant n_1$, i.e. $(x_\alpha, x_\beta) \in U \cap W \subset W$ where $\alpha, \beta \geqslant n_1$. Therefore $\{x_n\}$ is a Cauchy net as W is an arbitrary member of u. By the completeness of graph of f, $x_n \to p$ and $fx_n \to fp$ whence $p = fp$. This completes the proof.

It is obvious that the fixed point is not contractive as in Theorem 1. But in the following we see that it is so if X is a U-chainable uniform space.

Definition — A uniform space (X, u) is said to be U-chainable if for a $U \in u$ $x, y \in X$ there exists an integer k such that $(x, y) \in U^k$. The space is said to be well-chained if it is U-chainable for and every $U \in u$.

We now proceed to establish the contractiveness of the fixed point if the space in Theorem 2 is U-chainable.

We first prove that $(x, y) \in U$ and $x_n \to p \Rightarrow y_n \to p$. Let V be a symmetric member of u. We can find a $W \in u$ such that $W^2 \subset U \cap V$. Further, $f_2^n(U) \subset W$ for $n \geqslant m$, implying also $(x_n, y_n) \in W$ for $n \geqslant m$. Since $x_n \in W[p]$ for large n we have $(p, x_n) \in W$. Thus $(p, y_n) \in W^2 \subset V$, i.e. $y_n \in V[p]$ for large n. Hence $y_n \to p$.

Now since the space X is U-chainable, for $x, y \in X$ there exists a k such that $(x, y) \in U^k$, i.e. there exist points $x = z_0, z_1, z_2, \ldots, z_k = y$ such that $(z_i, z_{i+1}) \in U$, $0 \leqslant i \leqslant k-1$. Hence, by the above result, the iterates of these points converge to the same point provided the sequence of iterates of at least one point converges.

Remark : If the space is U-chainable then the assumption that $(x, fx) \in U$ for some $x \in X$ is not necessary in establishing the existence (and uniqueness) of the fixed point.

Since the space X is U-chainable, then $(x, fx) \in U^k$ for some integer k. Then $x = z_0, z_1, z_2, \ldots, z_k = fx$ are such that $(z_i, z_{i+1}) \in U$. If V is any member of u, then choose a member W of u such that $W^k \subset V \cap U$. Let $f_2^l(U) \subset W$ for some integer l. Then $(f^l z_i, f^l z_{i+1}) \in W, i = 0, 1, 2, \ldots, k-1$. Hence

$$(f^l z_0, f^l z_k) = (f^l x, f^{l+1} x) \in W^k \subset V \cap U \subset U.$$

Thus we can say that there exists a point a such that $(a, fa) \in U$. Now we may proceed as in Theorem 2 to establish the fixed point.

We now examine the above results in the light of the results derived by Knill (1965) who defines a function f of X into itself to be a contraction if for a $U \in u$, there is a closed member V of u such that

$$f_2(V) \subset \text{Int}\,(V) \subset U. \qquad \ldots(6)$$

Further, the self-map f is called a uniform contraction if for any member U of u there are members V and W of u such that

$$f_2(V) \circ W \subset V \subset U. \qquad \ldots(7)$$

If X is a compact connected Hausdorff space and f is a contraction, then, by Theorem 2.7 of Knill (1965), the filterbase $\{f^n(X)\}$ converges.

By the definition of contraction we can find a $U \in u$ such that

$$f_2(U) \subset \text{Int}\,(U) \subset U \subset X \times X. \qquad \ldots(8)$$

Since $\{f^n(X)\}$ converges, there exists an integer k such that $f^k(X) \times f^k(X) \subset V$, where V is any member of u. Hence $f_2^k(U) \subset V$. But, since $f_2(U) \subset U$, by (8) we have

$$f_2^{k+1}(U) \subset f_2^k(f_2(U)) \subset f_2^k(U) \subset V. \qquad \ldots(9)$$

In this manner we can show that $f_2^m(U) \subset V$ for $m \geqslant k$. Thus the conditions of Theorem 2 are satisfied and, therefore Corollary 2.6 of Knill (1965) follows from our results.

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