*Indian J. pure appl. Math.*, 12(4) : 452–458, April 1981

# ON $| \bar{N}, p_n |$ SUMMABILITY OF AN ASSOCIATED CONJUGATE SERIES OF A FOURIER SERIFS

G. C. HOTTA AND T. C. PADHI

*Post Graduate Department of Mathematics, Khallikote College, Berhampur* 760001, *Ganjam* (*Orissa*)

(*Received* 16 *June* 1980)

In the present paper we have not only generalized a theorem of Ray (1969) by extending it to $| \bar{N}, p_n |$ summability but also replaced the factor $\log (k/t)$ in his theorem by a more general function $Q(1/t)$.

§1. Let $\Sigma\, a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants real or complex, and

$$P_n = \sum_{\nu=0}^{n} p_\nu \neq 0, \quad \text{and} \quad P_{-1} = p_{-1} = 0.$$

The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu S_\nu = \frac{1}{P_n} \sum_{\nu=1}^{n} (P_n - P_{\nu-1})\, a_\nu \qquad \text{...(1.1)}$$

defines the sequence $\{t_n\}$ of $(\bar{N}, p_n)$ mean (Hardy 1949) of the sequence $\{S_n\}$, or of the series $\Sigma\, a_n$, generated by the sequence of coefficients $\{p_n\}$. The series $\Sigma\, a_n$ is said to be absolutely summable $(\bar{N}, p_n)$ or simply summable $| \bar{N}, p_n |$, if the sequence $\{t_n\}$ is of bounded variation, that is,

$$\sum_{n=1}^{\infty} | t_n - t_{n-1} | < \infty. \qquad \text{...(1.2)}$$

The conditions of regularity of the method of summability $(\bar{N}, p_n)$ defined by (1.1) are

$$\lim_{n\to\infty} | P_n | = \infty \qquad \text{...(1.3)}$$

and

$$\sum_{\nu=0}^{n} | p_\nu | = O( | P_n | ). \qquad \text{...(1.4)}$$

If $\{p_n\}$ is real and nonnegative, (1.4) is automatically satisfied, and then (1.3) is a necessary and sufficient condition for the regularity of the method.

§2. Let $f(t)$ be a periodic function with period $2\pi$ and integrable $L$ over $(-\pi, \pi)$. Let

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad \ldots(2.1)$$

be a Fourier series of $f(t)$, then the series conjugate to (2.1) at $t = x$ is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \text{ say.} \qquad \ldots(2.2)$$

We write $\psi(t) = \frac{1}{2} \{f(x + t) - f(x - t)\}$,

$$\Psi(t) = \int_t^{\pi} \frac{\psi(u)}{u}\, du = \psi_1(t)\, Q(T) \qquad \ldots(2.3)$$

where $Q(T)$, $T = (1/t)$, is a positive differentiable function tending to $+\infty$ as $t \to +0$, such that $Q'(T)$ is a positive decreasing function of $T$ tending to zero as $T \to +\infty$.

Denote

$$\triangle u_n = u_n - u_{n+1}.$$

Ray (1969) proved the following theorem on | $R$, $\log \omega$, 1 | summability of a series associated with the conjugate series of a Fourier series.

*Theorem A* — Let $\psi_1(t) \log \left(\frac{k}{t}\right) = \int_t^{\pi} \frac{\psi(u)}{u}\, du$, $k > \pi$.

If $\psi_1(t) \in BV(0, \pi)$ and $\int_0^{\pi} | \psi_1(t) | /t\,dt < \infty$, then

$$\sum \frac{B_n(x)}{\log (n + 1)} \text{ is summable } | R, \log \omega, 1 | .$$

In the above theorem we have observed a relation between the factors of $\psi_1(t)$ and $B_n(x)$.

In the present paper as a bid to generalize the above theorem we have replaced the factor $\log (k/t)$ by a general function $Q(1/t)$ and also extended it to | $\bar{N}, p_n$ | summability.

We prove the following:

*Theorem* — Let $\int_t^{\pi} \frac{\psi(u)}{u}\, du = \psi_1(t)\, Q(T)$

where $Q(T)$ defined as in (2.3).

If $\psi_1(t) \in BV(0, \pi)$, $\int_0^\pi \frac{|\psi_1(t)|}{t}\, dt < \infty$, and $\frac{TQ'(T)}{P_T}$ is bounded, then $\Sigma B_n(x) P_n^{-1}$ is summable $|\bar{N}, p_n|$ where $\{p_n\}$ is a sequence of nonnegative, and non-increasing numbers such that the sequence $\{Q(n)/P_n\}$ is bounded.

§3. We require the following lemmas for the proof of our theorem.

*Lemma* 1 — If $\{p_n\}$ is a sequence of nonnegative and non-increasing numbers, then $\{n/P_n\}$ is non-decreasing.

PROOF: Since $\{p_n\}$ is a nonnegative and non-increasing sequence,

$$P_n = \sum_{\nu=0}^{n} p_\nu \geqslant (n+1)\, p_n.$$

Now $$\Delta\left(\frac{n}{P_n}\right) = \frac{n}{P_n} - \frac{n+1}{P_{n+1}} = \frac{nP_{n+1} - (n+1)P_n}{P_n P_{n+1}} = \frac{np_{n+1} - P_n}{P_n P_{n+1}} \leqslant 0.$$

Hence the proof is complete.

*Lemma* 2 — If $\{p_n\}$ is a sequence of nonnegative and non-increasing numbers, then the sequence $\{p_n/P_n\}$ is non-increasing.

The result is obvious. We require the following estimates.

If $\{p_n\}$ is a sequence of nonnegative and non-increasing numbers, then

$$\left|\sum_{\nu=0}^{n} \sin \nu t\right| \leqslant \frac{A}{t} \qquad \ldots(3.1)$$

$$\left|\sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t\right| = \begin{cases} O(n^2 t) & \text{for all } t \\ O(t^{-1}) & \text{for } \geqslant n^{-1} \end{cases} \qquad \ldots(3.2)$$

PROOF: The estimates (3.1) and the first part of (3.2) are trivial. We prove the 2nd part of (3.2) using (3.1) and Abel's lemma.

We have

$$\left|\sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t\right| = \left|\sum_{\nu=1}^{n} \left(1 - \frac{p_\nu}{P_\nu}\right) \sin \nu t\right|$$

*(equation continued on p. 455)*

$$\leqslant \left| \sum_{\nu=1}^{n} \sin \nu t \right| + \left| \sum_{\nu=1}^{n} \frac{p_\nu}{P_\nu} \sin \nu t \right|$$

$$= O(t^{-1}) + O(t^{-1})$$

$$= O(t^{-1}).$$

§4. *Proof of the Theorem* — Using (1.1), we have

$$\sum_{n=1}^{\infty} \left| t_n - t_{n-1} \right| = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{n=1}^{n} P_{\nu-1} a_\nu \right|,$$

where $a_\nu = \frac{B_\nu(x)}{P_\nu}$ for our series.

$$B_n(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \sin nt \, dt$$

$$= \frac{2}{\pi} \int_0^\pi t \frac{\psi(t)}{t} \sin nt \, dt$$

$$= -\frac{2}{\pi} \int_0^\pi t \Psi'(t) \sin nt \, dt$$

$$= -\frac{2}{\pi} \int_0^\pi t \sin nt \frac{d}{dt} (\psi_1(t)\, Q(T)) \, dt$$

$$= -\frac{2}{\pi} \int_0^\pi t \sin nt \, dt \left[ Q(T) \frac{d}{dt} \psi_1(t) + \psi_1(t)\, Q'(T) \left( -\frac{1}{t^2} \right) \right]$$

$$= \frac{2}{\pi} \int_0^\pi Q'(T) \psi_1(t) \frac{\sin nt}{t} \, dt - \frac{2}{\pi} \int_0^\pi t Q(T) \sin nt \, d\psi_1(t).$$

Now,

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} P_{\nu-1} a_\nu \right| =$$

(*equation continued on p.* 456)

$$= \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} P_{\nu-1} \frac{2}{\pi} \frac{1}{P_\nu} \left[ \int_0^\pi Q'(T)\, \psi_1(t) \frac{\sin \nu t}{t}\, dt \right.\right.$$

$$\left.\left. - \int_0^\pi tQ(T) \sin \nu t\, d\psi_1(t) \right] \right|$$

$$\leqslant \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \int_0^\pi Q'(T)\, \psi_1(t) \frac{\sin \nu t}{t}\, dt \right|$$

$$+ \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \int_0^\pi tQ(T) \sin \nu t\, d\psi_1(t) \right|$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

The theorem will be established if we can show

$$\Sigma_1 = O(1) \quad \text{and} \quad \Sigma_2 = O(1).$$

Now

$$\sum\nolimits_1 = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \int_0^\pi Q'(T)\, \psi_1(t) \frac{\sin \nu t}{t}\, dt \right|$$

$$\leqslant \int_0^\pi \frac{|\psi_1(t)|}{t}\, dt\, Q'(T) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right|$$

Since by the hypothesis $\int_0^\pi |\psi_1(t)|\, t^{-1}\, dt < \infty$, to prove $\Sigma_1 = O(1)$, it is sufficient to show that

$$Q'(T) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right| = O(1)$$

uniformly for all $t$, $0 < t < \pi$.

Using the estimates (3.2), and Lemmas 1 and 2, we have

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right|$$

$$= \sum_{n<T} + \sum_{n \geqslant T}$$

(*equation continued on p.* 457)

$$= \sum_{n<T} \frac{p_n}{P_nP_{n-1}}\, O(n^2t) + \sum_{n\geqslant T} \frac{p_n}{P_nP_{n-1}}\, O(t^{-1})$$

$$= O(t) \sum_{n<T} \frac{n}{P_n} + O(T)\left(\frac{1}{P_T}\right)$$

$$= O\left(t\frac{T}{P_T}\right) O(T) + O\left(\frac{T}{P_T}\right) = O\left(\frac{T}{P_T}\right).$$

Hence, $$Q'(T) \sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right|$$

$$= Q'(T)\, O\left(\frac{T}{P_T}\right) = O\left(\frac{TQ'(T)}{P_T}\right) = O(1).$$

Again $$\sum\nolimits_2 = \sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \int_0^{\pi} tQ(T) \sin \nu t\, d\psi_1(t) \right|$$

$$\leqslant \int_0^{\pi} |\, d\psi_1(t)\, |\, tQ(T) \sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right|.$$

Since by the hypothesis $\int_0^{\pi} |\, d\psi_1(t)\, | < \infty$, to prove $\Sigma_2 = O(1)$, it is sufficient to show that

$$tQ(T) \sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \left| \sum_{\nu=1}^{n} \frac{p_{\nu-1}}{P_\nu} \sin \nu t \right| = O(1)$$

uniformly for all $t$, $0 < t < \pi$.

Now using the estimates (3.2) and, Lemmas 1 and 2,

$$\sum_{n=1}^{\infty} \frac{p_n}{P_nP_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right|$$

$$= \sum_{n<T} + \sum_{n\geqslant T}$$

$$= \sum_{n<T} \frac{p_n}{P_nP_{n-1}}\, O(n^2t) + \sum_{n\geqslant T} \frac{p_n}{P_nP_{n-1}}\, O\left(\frac{1}{t}\right).$$

(*equation continued on p.* 458)

$$= O(t) \sum_{n<T} \frac{n}{P_n} + \frac{O(T)}{P_T}$$

$$= O\left(t \frac{T}{P_T}\right) O(T) + O\left(\frac{T}{P_T}\right)$$

$$= O\left(\frac{T}{P_T}\right).$$

Hence

$$tQ(T) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{\nu=1}^{n} \frac{P_{\nu-1}}{P_\nu} \sin \nu t \right|$$

$$= O\left( tQ(T) \frac{T}{P_T} \right)$$

$$= O\left(\frac{Q(T)}{P_T}\right)$$

$$= O(1).$$

This completes the proof of our theorem.

It is easy to see that by taking $p_n = \frac{1}{n+1}$, and $Q(t^{-1}) = \log \frac{2\pi}{t}$, the result of Ray (1969) follows from our theorem. Besides the above case, our theorem yields many other results if $\{p_n\}$ be replaced by the following sequences

$$\left\{\frac{1}{(n+1)\log n}\right\}, \left\{\frac{1}{(n+1)\log n \log\log n}\right\},$$

$$\left\{\frac{1}{(n+1)\log n \log\log n \log\log\log n}\right\},$$

$$\cdots \left\{\frac{1}{(n+1)\log n \log\log n \ldots \log\log \ldots \log_{p-1} n(\log\log \ldots \log_p n)}\right\},$$

with suitable $Q(n)$.

## References

Hardy, G. H. (1949). Divergent Series. The Clarendon Press, Oxford.

Ray, B. K. (1969). On the absolute Reisz summability of a series associated with the conjugate series of a Fourier series. *Math. Student*, **37**, 91–99.