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INTEGRAL REPRESENTATION OF FUNCTIONS IN CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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For a fixed positive integer N let $\varepsilon = e^{2\pi i/N}$ and for an analytic function $f(z) = a_\mu z^\mu + a_{\mu+1}z^{\mu+1} + \ldots$ in the unit disc define

$$f_N(z) = \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-\mu k} f(\varepsilon^k z).$$

Let $S_N^*(\alpha)$, $K_N(\alpha)$ and $C_N(\alpha)$ be the classes of normalized analytic functions f in the unit disc for which

$$\text{Re}\,\{zf'(z)/f_N(z)\},\ \text{Re}\,(zf'(z))'/f_N'\{(z)\} \text{ and } \text{Re}\,\{zf'(z)/g_N(z)\}$$

for some $g \in S_N^*(0)$, respectively, are greater than $\alpha (0 \leqslant \alpha < 1)$. In this paper, coefficient bounds for these classes are discussed. Integral representations for functions in these classes are also obtained and these in turn are used to determine the closed-convex-hull of $C_N' = \{f' : f \in C_N(0)\}$.

1. INTRODUCTION

In this paper we consider the classes $S_N^*(\alpha)$, $K_N(\alpha)$ and $C_N(\alpha)$ consisting, respectively, of starlike, convex and close-to-convex functions of order α with respect to N symmetric points. These classes, which are obtained by generalising Sakaguchi's (1959) concept of starlike functions with respect to symmetric points, have been studied by Chand and Singh (1979) [also see Chand (1978)]. A normalized function

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

which is analytic in the unit disc $D = \{z : |z| < 1\}$ belongs to the class $S_N^*(\alpha)$, $K_N(\alpha)$ or $C_N(\alpha)$ according as it satisfies:

$$\text{Re}\left\{\frac{zf'(z)}{f_N(z)}\right\} > \alpha \qquad \ldots(1)$$

$$\text{Re}\left\{\frac{(zf'(z))'}{f'_N(z)}\right\} > \alpha \qquad \ldots(2)$$

or

$$\text{Re}\left\{\frac{zf'(z)}{q_N(z)}\right\} > \alpha \qquad \ldots(3)$$

for some $g \in S_N^*(0)$ where $0 \leqslant \alpha < 1$ and f_N is defined as follows.

Let N be a fixed positive integer and $\epsilon = e^{2\pi i/N}$ be the nth root of unity. For an analytical function

$$F(z) = A_k z^k + A_{k+1} z^{k+1} + \ldots \qquad \ldots(4)$$

in D define

$$F_N(z) = \frac{1}{N} \sum_{j=0}^{N-1} \epsilon^{-kj} F(\epsilon^j z) \qquad \ldots(5)$$

$$= A_k z^k + A_{N+k} z^{N+k} + A_{2N+k} z^{2N+k} + \ldots .$$

Thus for the normalized function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

we have $f_N(z) = z + \sum_{p=1}^{\infty} a_{pN+1} z^{pN+1}$.

Writing

$$\delta_n = \delta(n, N)$$

$$= \begin{cases} 1 \text{ if } n = pN + 1 \text{ for } p = 0, 1, 2 \ldots \\ 0 \text{ otherwise} \end{cases} \qquad \ldots(6)$$

we get $f_N(z) = z + \sum_{n=2}^{\infty} \delta_n a_n z^n$. We refer to the functions in these three classes as N-starlike, N-convex and N-close-to-convex functions of order α, respectively. For the particular case $\alpha = 0$ let the above mentioned three classes be denoted by S_N^*, K_N and C_N, respectively. Note that for $N = 1$ we get $f_N(z) = f(z)$ and the classes $S_N^*(\alpha)$, $K_N(\alpha)$ and $C_N(\alpha)$ become the usual classes $S^*(\alpha)$ $K(\alpha)$ and $C(\alpha)$ of starlike, convex and close-to-covex functions of order α, respectively. Also for $N = 2$ and $\alpha = 0$ the requirement (1) is equivalent to the condition for starlikeness with respect to symmetric points due to Sakaguchi (1959). It may be seen that $f \in S_N^*(\alpha)$ implies that

$f_N \in S_N^*(\alpha)$ and similar results hold for the classes $K_N(\alpha)$ and $C_N(\alpha)$. Also $f \in K_N(\alpha)$ if, and only if, $zf' \in S_N^*(\alpha)$. Further $K_N(\alpha) \subset S_N^*(\alpha) \subset C_N(\alpha) \subset C(\alpha)$ where $C(\alpha)$ is the class of normalised close-to-convex functions of order α.

In this paper we obtain coefficient bounds for the three classes under consideration and our result is a modification of that obtained earlier by Chand and Singh. Integral representations for functions in the classes C_N and S_N^* are also obtained and these in turn are used to obtain the closed-convex-hull of $C'_N = \{f' : f \in C_N\}$.

2. Coefficient Bounds

In this section coefficient bounds for the classes $S_N^*(\alpha)$, $K_N(\alpha)$ and $C_N(\alpha)$ are discussed. Bounds for these classes for the particular case $\alpha = 0$ have been considered by Chand (1978). The generalised result is as follows:

Theorem 2.1 — Let $f(z) = z + a_2 z^2 + \dots \in S_N^*(\alpha)$. Then

$$p^2 N^2 \mid a_{pN+1} \mid^2 \leqslant 4(1-\alpha)^2 + 2 \sum_{k=1}^{p-1} [(2-\alpha)(kN+1) - 2\alpha(1-\alpha)] \mid a_{kN+1} \mid^2 \quad \dots(7)$$

for $p = 1, 2, 3, \dots$, and

$$(pN+m)^2 \mid a_{pN+m} \mid^2 \leqslant 4(1-\alpha)^2 + 2 \sum_{k=1}^{p} [(2-\alpha)(kN+1) - 2\alpha(1-\alpha)] \mid a_{kN+1} \mid^2 \quad \dots(8)$$

for $m = 2, 3, \dots, N$ and $p = 0, 1, 2, 3, \dots$.

These inequalities are sharp.

Proof : Now $\operatorname{Re}\left\{\frac{zf'(z)}{f_N(z)}\right\} > \alpha$. Hence we may write

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f_N(z)} - \alpha\right) = \frac{1+w(z)}{1-w(z)}$$

for some analytic function $w(z)$ in $|z| < 1$ with $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$. Cross multiplication and rearrangement yields

$$zf'(z) - f_N(z) = [zf'(z) + (1-2\alpha) f_N(z)] \, w(z). \quad \dots(9)$$

But $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \; f_N(z) = z + \sum_{n=2}^{\infty} \delta_n a_n z^n$$

where δ_n is as defined in (6). Then (9) becomes

$$\sum_{n=2}^{\infty} (n - \delta_n)\, a_n Z^n = \left[2(1-\alpha)\, z + \sum_{n=2}^{\infty} (n + (1-2\alpha)\, \delta_n)\, a_n z^n\right] w(z). \quad \ldots(10)$$

Let $w(z) = \sum_{n=1}^{\infty} w_n z^n$. On comparing the coefficients of z^n on both sides of (10), we have for $n \geqslant 2$

$$(n - \delta_n)\, a_n = 2(1-\alpha)\, w_{n-1} + \sum_{k=2}^{n-1} (k + (1-2\alpha)\, \delta_k)\, a_k w_{n-k}. \quad \ldots(11)$$

This shows that the coefficient a_n depends upon the coefficients $a_2, a_3, \ldots, a_{n-1}$ and w_n's. Therefore, by a suitable choice of A_k's, (10) may be rewritten as

$$\sum_{k=2}^{n} (k - \delta_k)\, a_k z^k + \sum_{k=n+1}^{\infty} A_k z^k$$

$$= \left[2(1-\alpha)\, z + \sum_{k=2}^{n-1} (k + (1-2\alpha)\, \delta_k)\, a_k z^k\right] w(z) \quad \ldots(12)$$

where $n \geqslant 2$. Considering the square of the modulus on both sides and integrating over $|z| = r < 1$, we get

$$\sum_{k=2}^{n} (k - \delta_k)^2\, |a_k|^2\, r^{2k} + \sum_{k=n+1}^{\infty} |A_k|^2\, r^{2k}$$

$$\leqslant 4(1-\alpha)^2\, r^2 + \sum_{k=2}^{n-1} (k + (1-2\alpha)\, \delta_k)^2\, |a_k|^2\, r^{2k} \quad \ldots(13)$$

since $|w(z)| < 1$. Further, on letting $r \to 1$ this gives

$$\sum_{k=2}^{n} (k - \delta_k)^2\, |a_k|^2 \leqslant 4(1-\alpha)^2 + \sum_{k=2}^{n-1} [k + (1-2\alpha)\, \delta_k]\, |a_k|^2$$

or equivalently

$$(n - \delta_n)^2\, |a_n|^2 \leqslant 4(1-\alpha)^2 + \sum_{k=2}^{n-1} [(k + (1-2\alpha)\, \delta_k)^2 - (k - \delta_k)^2]\, |a_k|^2$$

(*equation continued on p.* 463)

$$= 4(1-\alpha)^2 + 2\sum_{k=2}^{n-1} [k + (1-\alpha)(k-2\alpha)]\,\delta_k\,|a_k|^2 \quad \ldots(14)$$

in view of the equality $\delta_k = (\delta_k)^2$, where $n \geqslant 2$. The required result now follows in view of the fact that $\delta_k = 1$ if $k = pN + 1$ for $p = 1, 2, 3 \ldots$ and $\delta_k = 0$ for all other values.

The inequalities in the above theorem are attained for the normalised function f given by

$$f'(z) = \frac{1 + (1-2\alpha)\,z}{(1-z)(1-z^N)^{(2-2\alpha)/N}}. \quad \ldots(15)$$

The following results follow from the inequalities (7) and (8) in view of the fact that $f \in K_N(\alpha)$ if, and only if $zf' \in S_N^*(\alpha)$.

Theorem 2.2 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_n(\alpha)$. Then

$$[pN(pN+1)]^2\,|a_{pN+1}|^2 \leqslant 4(1-\alpha)^2 + 2\sum_{k=1}^{p-1} [(2-\alpha)(kN+1) - 2\alpha(1-\alpha)]\,(kN+1)^2\,|a_{kN+1}|^2 \quad \ldots(16)$$

for $p = 1, 2, 3, \ldots$, and

$$(pN+m)^4\,|a_{pN+m}|^2 \leqslant 4(1-\alpha)^2 + 2\sum_{k=1}^{p} [(2-\alpha)(kN+1) - 2\alpha(1-\alpha)]\,(kN+1)^2\,|a_{kN+1}|^2 \quad \ldots(17)$$

for $p = 0, 1, 2, \ldots$ and $m = 2, 3, \ldots, N$.

For $\alpha = 0$ the above theorems give the coefficient bounds for the classes S_N^* and K_N determined earlier by Chand (1978). The following two results are modified form of his results in the sense that the bounds are now available for each coefficient explicitly.

Corollary 2.3 — If $f(z) = z + \sum_{n=2}^{\infty} a_n z_n \in S_N^*$, then

$$|a_{pN+1}| \leqslant \frac{1}{p!}\prod_{k=0}^{p-1}\left(\frac{2}{N} + k\right) = A_{pN+1} \quad \ldots(18)$$

for $p = 1, 2, 3, \ldots$, and

$$|a_{pN+m}| \leqslant \frac{N}{(pN+m)\,p!}\prod_{k=0}^{p}\left(\frac{2}{N} + k\right) = \frac{pN+2}{pN+m}\,A_{pN+1} \quad \ldots(19)$$

for $p = 0, 1, 2, \ldots, m = 2, 3, \ldots, N$ where A_{pN+1} is the coefficient of z^{pN+1} in the expansion of $z/(1 - z^N)^{2/N}$.

PROOF : For $\alpha = 0$, (7) becomes

$$p^2N^2 \mid a_{pN+1} \mid^2 \leqslant 4 + 4 \sum_{k=0}^{p-1} (kN + 1) \mid a_{kN+1} \mid^2 \text{ for } p = 1, 2, \ldots. \quad \ldots(20)$$

Putting $p = 1$ we get $N^2 \mid a_{N+1} \mid^2 \leqslant 4$, i.e.

$$\mid a_{N+1} \mid \leqslant \frac{2}{N}. \quad \ldots(21)$$

Again for $p = 2$, (20) gives

$$4N^2 \mid a_{2N+1} \mid^2 \leqslant 4 + 4(N + 1) \mid a_{N+1} \mid^2$$

and using (21) we obtain

$$\mid a_{2N+1} \mid \leqslant \frac{2(2 + N)}{2!\, N^2}. \quad \ldots(22)$$

Similarly putting $p = 3$ in (20) and using (21) and (22) we have

$$\mid a_{3N+1} \mid \leqslant \frac{2(2 + N)(2 + 2N)}{3!\, N^3}. \quad \ldots(23)$$

Generalizing, we have

$$\mid a_{pN+1} \mid \leqslant \frac{2(2 + N) \ldots (2 + \overline{p - 1}N)}{p!\, N^p} = A_{pN+1}$$

and this gives (18).

Further putting $\alpha = 0$ in (19) we have

$$(pN + m)^2 \mid a_{pN+m} \mid^2 \leqslant 4 + 4 \sum_{k=1}^{p} (kN + 1) \mid a_{kN+1} \mid^2 \quad \ldots(24)$$

for $p = 0, 1, \ldots$ and $m = 2, 3, 4, \ldots, N$. For $p = 0$, this yields for $m = 2, 3, \ldots, N$

$$\mid a_m \mid \leqslant \frac{2}{m}. \quad \ldots(25)$$

For $p = 1$, (24) and (21) will give

$$\mid a_{N+m} \mid \leqslant \frac{2}{N} \frac{2 + N}{m + N}. \quad \ldots(26)$$

Similarly for $p = 2$, we may verify that

$$| a_{2N+m} | \leqslant \frac{2(2+N)}{2!\,N^2} \frac{2+2N}{m+2N}. \qquad ...(27)$$

Generalizing this we obtain

$$| a_{pN+m} | \leqslant \frac{2+pN}{m+pN} \frac{2(2+N)\ldots(2+\overline{p-1}N)}{n!\,N^p}$$

which is equivalent to (19).

Corollary 2.4 — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_N$, then for $p = 1, 2, \ldots,$

$$| a_{pN+1} | \leqslant \frac{1}{(pN+1)\,p!} \prod_{k=0}^{p-1} \left(\frac{2}{N} + k\right) = \frac{1}{pN+1} A_{pN+1} \qquad ...(28)$$

and for $p = 0, 1, \ldots; m = 2, 3, \ldots, N$

$$| a_{pN+m} | \leqslant \frac{N}{(pN+m)^2\,p!} \prod_{k=0}^{p} \left(\frac{2}{N} + k\right) = \frac{pN+2}{(pN+m)^2} A_{pN+1}. \qquad ...(29)$$

PROOF : The proof may be completed as in the above Corollary.

Chand (1978) has shown that eqns. (20) and (24) are valid for the class C_N and consequently the explicit bounds given by (18) and (19) hold for the larger class C_N as well.

3. CLOSED-CONVEX-HULL OF C'_N

In this section we determine the closed-convex-hull of the class C'_N consisting of functions f' for which f is N-close-to-convex. For this purpose we need the following lemma which modifies Theorem 1, Corollary 1 and Theorem 2 of Brickman *et al.* (1973) and the proof of which is also based upon their method.

Lemma 3.1 — Let X be the unit circle $\{x : | x | = 1\}$, IP the set of all probability measures on X and p a positive real number. For $z \in D$ and $\mu \in IP$ write

$$f_\mu(z) = \int_X \frac{d\mu(x)}{(1 - x^N z^N)^p}. \qquad ...(30)$$

(This determines a function f_μ on D.) Let $T_p = \{f_\mu : \mu \in IP\}$

Then

(i) Product of a function in T_p with a function in T_q belongs to T_{p+q} where $p, q > 0$, i.e.

$T_p \circ T_q \subset T_{p+q}$.

(ii) If $0 < p < q$, then $T_p \subset T_q$.

(iii) For $\mu \in IP$ there exists $\lambda \in IP$ such that

$$\exp\left[\int_X -p \log(1 - x^N z^N)\, d\mu(x)\right] = \int_X \frac{d\lambda(x)}{(1 - x^N z^N)^p}. \qquad ...(31)$$

PROOF : In view of Fubini's theorem, the product

$$\int_X \frac{d\mu(x)}{(1 - x^N z^N)^p} \cdot \int_X \frac{d\lambda(y)}{(1 - y^N z^N)^q} = \int_{X^2} \frac{d(\mu x \lambda)}{(1 - x^N z^N)^p (1 - y^N z^N)^q}$$

where $\mu x \lambda$ is a probability measure on $X^2 = X \times X$. But each of the classes T_p is a closed convex family, in view of Theorem 1 of Brickman *et al.* (1971). Hence it is enough to show that the integrand

$$\frac{1}{(1 - x^N z^N)^p (1 - y^N z^N)^q} \quad \text{belongs to} \quad T_{p+q}.$$

This is obvious if $x^N = y^N$. Now assume that $x^N \neq y^N$ and write

$$c(t) = t x^N + (1 - t)\, y^N$$

where $0 \leqslant t \leqslant 1$. Obviously $|\, c(t) \,| \leqslant 1$. We may apply Lemma 1 of Brickman *et al.* (1971) to get

$$(1 - x^N z^N)^{-p} (1 - y^N z^N)^{-q}$$

$$= \frac{\Gamma(p + q)}{\Gamma(p)\, \Gamma(q)} \int^1 t^{p-1} (1 - t)^{q-1} [1 - c(t)\, z^N]^{-p-q}\, dt \qquad ...(32)$$

But $\frac{\Gamma(p + q)}{\Gamma(p)\, \Gamma(q)} t^{p-1}(1 - t)^{q-1}\, dt$ determines a probability measure on the interval [0, 1] so that it is enough to show that

$$[1 - c(t)\, z^N]^{-p-q} \in T_{p+q} \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

This is obvious if $t = 0$ or 1. For $0 < t < 1$, we may use Poisson's integral formula to get

$$[1 - c(t)\, z^N]^{-p-q} = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left\{\frac{e^{i\theta} + c(t)^{1/N}}{e^{i\theta} - c(t)^{1/N}}\right\} (1 - e^{i\theta} z^N)^{-p-q}\, d\theta \quad ...(33)$$

This integral belongs to T_{p+q} as $\operatorname{Re}\left\{\frac{e^{i\theta} + c(t)^{1/N}}{e^{i\theta} - c(t)^{1/N}}\right\} d\theta$ determines a measure on X.

Hence (i) is proved.

Next for $|z| < 1$ and $0 < p < q$ we see that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1 - e^{iN\theta} z^N)^{q-p}} = 1. \qquad \ldots(34)$$

This, together with part (i) of the lemma, shows that $T_p \subset T_q$.

To prove (iii) let $\mu \in IP$. Then there exists a net $\{\mu_\alpha\}$ converging to μ such that each μ_α is a measure on X determined by a finite set of point masses. Consequently the integral $\int_X -p \log(1 - x^N z^N)\, d\mu(x)$ is the locally uniform limit of integrals of the form $\int_X -p \log(1 - x^N)\, d\mu_\alpha$. Now if the measure μ_α corresponds to the measure determined by the masses w_i's at the points x_i's for $i = 1, 2, \ldots$, then this last integral is a sum of the form $\Sigma - p w_i \log(1 - x_i^N z^N)$ where $|x_i| = 1, w_i > 0$ and $\Sigma w_i = 1$. Thus $\exp[\int - p \log(1 - x^N z^N)\, d\mu]$ is approximated by the products of the form $\prod_i (1 - x_i^N z^N)^{-pw_i}$ which belongs to T_p in view of (i) above. But T_p is a closed subset of IA, the space of all analytic functions on D endowed with the topology of uniform convergence on compact subsets of D. Hence the limit of such products also belongs to T_p i.e. $\exp[\int - p \log(1 - x^N z^N)\, d\mu] \in T_p$. Thus (iii) follows.

Theorem 3.2 — Let X be the unit circle, X^2 the torus $\{(x, y) : |x| = 1 = |y|\}$ and IM the set of all probability measures on X^2. For $z \in D$ and $|x| = 1 = |y|$ define

$$k(z, x, y) = \frac{1 + xz}{(1 - xz)(1 - y^N z^N)^{2/N}} \qquad \ldots(35)$$

and for

$$\mu \in IM \text{ let } f_\mu(z) \int_{X^2} k(z, x, y)\, d\mu. \qquad \ldots(36)$$

[Thus $k(z, x, y)$ and $f_\mu(z)$ determine functions on D]

Let $C'_N = \{f' : f \in C_N\}$ where C_N is the set of all normalized N-close-to-convex functions. Then closed-convex-hull of C'_N is the class $IF = \{f_\mu : \mu \in IM\}$.

Proof : Since X^2 is a compact Hausdorff space, we may apply Theorem 1 of Brickman *et al.* (1971) to conclude that IF is a compact subset of IA and is the closed-convex-hull of the family of functions of the form $k(z, x, y)$ where $|x| = 1 = |y|$. We may verify that $k(z, x, y) \in C'_N$ for each x, y with $|x| = 1 = |y|$. Thus $IF \subset$ clco C'_N where clco denotes the closed-convex-hull in IA.

To prove the reverse inclusion, viz. $IF \supset$ clco. C'_N, let $f \in C_N$. By definition, there exists $g \in S^*_N$ such that $\operatorname{Re}\left\{\frac{zf'(g)}{g_N(z)}\right\} > 0$ for $z \in D$. Then by Herglotz's theorem there exists some probability measure λ on the unit circle such that

$$\frac{zf'(z)}{g_N(z)} = \int_X \frac{1+xz}{1-xz}\, d\lambda(x). \qquad \ldots(37)$$

Also $g \in S^*_N$ implies that $\operatorname{Re} zg'(z)/g_N(z) > 0$ for $z \in D$ and we have

$$\frac{zg'(z)}{g_N(z)} = \int_X \frac{1+yz}{1-yz}\, d\nu(y) \qquad \ldots(38)$$

for some probability measure ν on X. Changing z to $\epsilon^k z$ in (38) and adding the N equations for $k = 0, 1, \ldots, N-1$, we find that

$$\frac{zg'_N(z)}{g_N(z)} = \int_X \left[\sum_{k=0}^{N-1} \frac{1+\epsilon^k yz}{1-\epsilon^k yz}\right] d\nu(y) = \int_X \frac{1+y^N z^N}{1-y^N z^N}\, d\nu(y).$$

Integration with respect to z and simplification yields

$$g_N(z) = z \exp\left[-\frac{2}{N}\int_X \log(1-y^N z^N)\, d\nu(y)\right]$$

$$= \int_X \frac{z}{(1-y^N z^N)^{2/N}}\, d\eta(y) \qquad \ldots(39)$$

where η is a probability measure on X, the existence of which is assured by part (iii) of Lemma 3.1. Then combining (37) and (39) we get

$$f'(z) = \int_X \frac{1+xz}{1-xz}\, d\lambda(x) \, . \int_X \frac{z}{(1-y^N z^N)^{2/N}}\, d\eta(y)$$

$$= \int_{X^2} \frac{1+xz}{(1-xz)(1-y^N z^N)^{2/N}}\, d(\lambda x \eta)$$

by Fubini's theorem, where $\lambda x \eta$ is a probability measure on IM.

This shows that $f'(z) \in IF$ and from this we conclude that clco. $C'_N \subset IF$. This completes the proof of the theorem.

The following corollary is an easy consequence of the theorem.

Corollary 3.3 — If $f \in C_N$ then there exists a probability measure μ on X^2 such that

$$f'(z) = \int_{X^2} \frac{1 + xz}{(1 - xz)(1 - y^N z^N)^{2/N}} \, d\mu \qquad \ldots(40)$$

Theorem 3.4 — Let f be a normalized analytic function on D. Then $f \in S_N^*$ if, and only if,

$$f'(z) = \int_{X^2} \frac{1 - xz}{(1 - xz)(1 - y^N z^N)^{2/N}} \, d(\lambda x \mu) \qquad \ldots(41)$$

where λ, μ are some probability measures on the unit circle X, X^2 is the torus $\{(x, y) : |x| = 1 = |y|\}$ and λ, μ satisfy

$$\exp\left[\int_X -\frac{2}{N} \log(1 - x^N z^N) \, d\lambda(x)\right] = \int_X \frac{d\mu(x)}{(1 - x^N z^N)^{2/N}}. \qquad \ldots(42)$$

PROOF: Let $f(z) = z + a_2 z^2 + \ldots \in S_N^*$. Then $\operatorname{Re}\{zf'(z)/f_N(z)\} > 0$ for $z \in D$ and by Herglotz's theorem we have

$$\frac{zf'(z)}{f_N(z)} = \int_X \frac{1 + xz}{1 - xz} \, d\lambda(x) \qquad \ldots(43)$$

for some probability measure λ on X. Now changing z to $\epsilon^k z$ and adding the N equations obtained from (43) by putting $k = 0, 1, \ldots, N - 1$, we get

$$\frac{zf_N'(z)}{f_N(z)} = \frac{1}{N} \int_X \left(\sum_{k=0}^{N-1} \frac{1 + \epsilon^k xz}{1 - \epsilon^k xz} \right) d\lambda = \int_X \frac{1 + x^N z^N}{1 - x^N z^N} \, d\lambda.$$

Then integration with respect to z yields

$$f_N(z) = z \exp\left[\int_X -\frac{2}{N} \log(1 - x^N z^N) \, d\lambda(x)\right].$$

By Lemma 3.1 (iii), there exists a probability measure μ on X such that

$$\exp\left[\int_X -\frac{2}{N} \log(1 - x^N z^N) \, d\lambda(x)\right] = \int_X \frac{d\mu(x)}{(1 - x^N z^N)^{2/N}}.$$

This gives

$$f_N(z) = \int_X \frac{z}{(1 - x^N z^N)^{2/N}} \, d\mu(x)$$

and hence from (43)

$$f'(z) = \int_X \frac{1+xz}{1-xz}\, d\lambda \,.\, \int_X \frac{1}{(1-x^N z^N)^{2/N}}\, d\mu$$

$$= \int_{X^2} \frac{1+xz}{(1-xz)\,(1-y^N z^N)^{2/N}}\, d(\lambda x \mu)$$

by Fubini's theorem, where λ, μ satisfy (42). Thus the necessity part of the theorem is proved.

Conversely let f' be given by (41) where λ, μ are related by (42). We may write

$$f'(z) = \int_X \frac{1+xz}{1-xz}\, d\lambda(x) \,.\, \int_X \frac{d\mu(x)}{(1-y^N z^N)^{2/N}}. \qquad \ldots(44)$$

Then
$$f'_N(z) = \frac{1}{N} \sum_{k=0}^{N-1} f'(\epsilon^k z)$$

$$= \left[\int_X \frac{d\mu(x)}{(1-y^N z^N)^{2/N}}\right] \cdot \frac{1}{N} \left[\int_X \sum_{k=0}^{N-1} \frac{1+\epsilon^k xz}{1-\epsilon^k xz}\, d\lambda(x)\right]$$

$$= \left[\exp \int_X -\frac{2}{N} \log\,(1-x^N z^N)\, d\lambda(x)\right] \cdot \left[\int_X \frac{1+x^N z^N}{1-x^N z^N}\, d\lambda(x)\right]$$

$$= F_1(z)\, F_2(z)$$

where

$$F_1(z) = \exp \int_X -\frac{2}{N} \log\,(1-x^N z^N)\, d\lambda(x)$$

and

$$F_2(z) = \int_X \frac{1+x^N z^N}{1-x^N z^N}\, d\lambda(x).$$

Differentiation with respect to z under the integral sign yields

$$z \frac{d}{dz} (\log F_1(z)) = \int_X \frac{2x^N z^N}{1-x^N z^N}\, d\lambda(x) = F_2(z) - 1$$

or

$$F_2(z) = 1 + z \frac{d}{dz} (\log F_1(z)) = 1 + zF'_1(z)/F_1(z).$$

On substituting in the above expression for $f'_N(z)$, we get

$$f'_N(z) = F_1(z) + zF'_1(z)$$

and this in turn leads to

$$f_N(z) = zF_1(z) = z\int_X \frac{d\mu(x)}{(1 - y^N z^N)^{2/N}}. \qquad ...(45)$$

Hence

$$\frac{zf'(z)}{f_N(z)} = \int_X \frac{1 + xz}{1 - xz}\, d\lambda(x)$$

and we conclude that $\text{Re}\left\{\frac{zf'(z)}{f_N(z)}\right\} > 0$ for $|z| < 1$.

This completes the proof of the theorem.

Further using the fact that $f \in K_N$ if and only if $zf' \in S^*_N$ this representation may be used to obtain a representation for functions in K_N.

The form of the above representations for f' when f belongs to one of the classes S^*_N, K_N or C_N may be useful in determining closed-convex-hulls of these classes and also their extreme points. For example, in view of Theorem $I(d)$ of Brickman *et al.* (1971) the only possible extreme-points of clco. C'_N are functions of the form

$$z \to k(z, x, y) = \frac{1 + xz}{(1 - xz)(1 - y^N z^N)^{2/N}}$$

and this observation may lead to the determination of extreme-points of clco. C_N and clco. S_N. Extreme points for these classes are completely known for the particular case $N = 1$ but the determination of the same for all values of N is an open problem.

REFERENCES

Brickman, L., MacGregor, T. H., and Wilken, D. R. (1971). Convex hulls of some classes of families of univalent functions. *Trans. Am. math. Soc.*, **156**, 91–108.

Brickman, L., Hellenbeck, D. J., MacGregor, T. H., and Wilken, D. R. (1973). Convex hulls and extreme points of families of starlike and convex functions. *Trans. Am. math. Soc.*, **185**, 413–28.

Chand, Rattan (1978). Some aspects of functions analytic in the unit disc. Ph.D. Thesis, I.I.T. Kanpur.

Chand, Rattan, and Singh, P. (1979). On certain Schlicht mappings. *Indian J. pure appl. Math.*, **10**, 1167–74.

Sakaguchi, K. (1959). On a certain univalent mapping. *J. Math. Soc. Japan*, **11**, 72–80.

Singh, P. P. (1979). A study of some classes of analytic functions in the unit disc. Ph.D. Thesis, I.I.T. Kanpur.