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ON A UNIFIED CLASS OF POLYNOMIALS*

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This paper deals with the study of the polynomials $p_n^{(\alpha)}(x)$, satisfying the functional relation

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x), \ n = 0, 1, 2, \ldots;$$

where $\Delta_\alpha \{f(\alpha)\} = f(\alpha + 1) - f(\alpha)$. Several properties notably generating function, recurrence relation, expansions and some characterization theorems have been discussed. Algebraic structure on the set of polynomials $\{p_n^{(\alpha)}(x)\}$ has also been studied.

1. INTRODUCTION

Appell (1880) introduced and studied the polynomials $\{P_n(x)\}$ defined by

$$\frac{d}{dx} P_n(x) = P_{n-1}(x); \ n = 1, 2, \ldots . \qquad \ldots(1.1)$$

Sheffer (1939) and Steffensen (1941) generalized Appell sets by considering a linear differential operator of infinite order with constant coefficients

$$L(D) = \sum_{k=0}^{\infty} C_k D^{k+1}, \quad C_0 \neq 0 \qquad \ldots(1.2)$$

as a generalization of the differential operator D. Both consider the polynomial set $\{P_n(x)\}$ which satisfy

$$L(D)\, p_n(x) = p_{n-1}(x). \qquad \ldots(1.3)$$

Later, Ožegov (1964) has generalized Appell sets in a different direction. He studied polynomial sets which have the property

$$D^r p_n(x) = p_{n-r}(x), \quad (n = r, r + 1, \ldots) \qquad \ldots(1.4)$$

where r is a (fixed) positive integer.

Recently, Al-Salam and Verma (1970) studied the class of polynomials $\{p_n(x)\}$ with the property

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$$J(D)\, p_n(x) = p_{n-r}(x), \quad (n = r, r+1, \ldots) \qquad \ldots(1.5)$$

where r is a fixed positive integer and

$$J(D) = \sum_{k=0}^{\infty} a_k D^{k+r}, \quad a_0 \neq 0$$

in which a_k $(k \geqslant 0)$ is independent of x.

Obviously the above class contains both Ožegov class $A^{(r)}$ and Sheffer A-type zero polynomials as special cases.

While Truesdell (1948) centered his investigations around the functional equation

$$\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1) \qquad \ldots(1.6)$$

one finds that a good number of classical polynomials such as Laguerre (Rainville 1960), generalized Laguerre (Chandel 1969, Mittal 1971), modified Laguerre

$$f_n^{(\alpha)}(x) = (-)^n L_n^{(-\alpha-n)}(x),$$

Tricomi (1951), Meixner [Erdélyi *et al.* 1953, (13), p. 225], Charlier [Erdélyi *et al.* 1953, (6), p. 226] (for $a = -1$ and $x = -x$) and the polynomials $g_n^{(k)}(x)$ known as the kth Cesáro mean [Erdélyi *et al.* 1955, (3), p. 245], $g_n(x)$ studied by Wright [Erdélyi *et al.* 1955, (14), p. 238] and $p_n^{(\alpha)}(x)$, defined as:

$$(1-t)^{-k-1}(1-xt)^{-1} = \sum_{n=0}^{\infty} g_n^{(k)}(x)\, t^n; \quad k = 0, 1, 2, \ldots; \qquad \ldots(1.7)$$

$$(1-t)^{\beta}\, \Phi(t) \exp\left(x/(1-t)\right) = \sum_{n=0}^{\infty} g_n(x)\, t^n, \quad (\,|\,t\,| < 1) \qquad \ldots(1.8)$$

$$p_n^{(\alpha)}(x) = \frac{(x+\alpha)_n}{n!} \qquad \ldots(1.9)$$

satisfy relations of the type:

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x) \qquad \ldots(1.10)$$

or

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = n p_{n-1}^{(\alpha+1)}(x) \qquad \ldots(1.11)$$

where $\Delta_\alpha\{f(\alpha)\} = f(\alpha+1) - f(\alpha)$ and $p_n^{(\alpha)}(x)$ is a polynomial of degree n in x such that $p_0^{(\alpha)}(x) = 1$.

It may be remarked here that sets $p_n^{(\alpha)}(x)$ such that

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x)$$

or

$$\Delta_x \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x)$$

are essentially different. This is obvious from the simple illustration,

$$\Delta_\alpha \{L_n^{(\alpha)}(x)\} = L_{n-1}^{(\alpha+1)}(x)$$

but

$$\Delta_x \{F_n(x, \alpha)\} = L_{n-1}^{(\alpha+1)}(x)$$

where

$$F_n(x, \alpha) = \frac{-(1+\alpha)_n}{n!} \sum_{k=1}^{n} \frac{(-n)_k B_k(x)}{(1+\alpha)_k k!}$$

with $B_k(x)$, a Bernoulli polynomial of degree k, and $L_n^{(\alpha)}(x)$, the classical Laguerre polynomial.

In addition to the above mentioned polynomials, there are other polynomials too, which satisfy (1.10) or (1.11), with suitable combinations. As illustrations, we mention below three such important cases:

$$\Delta_p \left\{\frac{(p)_n}{(a+1)_n} H_n^{(a,b-n)}(q, p; x)\right\} = \frac{(p+1)_{n-1}}{(a+1)_{n-1}} H_{n-1}^{(a,b-n+1)}(q, p+1; x) \qquad \ldots(1.12)$$

$$\Delta_b \{(2/(x-1))^n P_n^{(a-n,b)}(x)\} = (2/(x-1))^{n-1} P_{n-1}^{(a-n+1,b+1)}(x) \qquad \ldots(1.13)$$

and

$$\Delta_a \left\{\frac{(b/x)^n}{n!} y_n(a-n, b; x)\right\} = \frac{(b/x)^{n-1}}{(n-1)!} y_{n-1}(a-n+2, b; x) \qquad \ldots(1.14)$$

where $H_n^{(a,b)}(p, q; x)$, $P_n^{(a,b)}(x)$ and $y_n(a, b; x)$ are generalized Rice's (Khandekar 1964), Jacobi (Rainville 1960) and generalized Bessel (Krall and Frink 1949) polynomials respectively.

Sheffer (1939) while studying the Appell polynomials has indicated the study of those polynomials which satisfy the functional relation

$$\Delta_x \{P_n(x)\} = P_{n-1}(x). \qquad \ldots(1.15)$$

Obviously (1.10) and (1.15) are quite different in nature. A close examination of the works of Appell, Truesdell and other mathematicians naturally suggests a study of the properties and the nature of the polynomials $p_n^{(\alpha)}(x)$ defined by (1.10) or (1.11).

In this paper an attempt has been made to study the sets of polynomials which satisfy a relation of the type (1.10). The finite difference and shift operators are the prime tools in our analysis.

2. Some Theorems on Polynomials $p_n^{(\alpha)}(x)$

Theorem 1 — A necessary and sufficient condition that $p_n^{(\alpha)}(x)$ satisfies the relation $\Delta_\alpha\{p_n^{(\alpha)}(x)\} = p_n^{(\alpha+1)}(x)$ is that

$$(1-t)^{-\alpha} F(x, t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n \qquad \ldots(2.1)$$

where $F(x, t)$ is a periodic function of period unity in α.

Proof : Operating both sides of (2.1) by Δ_α, we get

$$\sum_{n=1}^{\infty} t^n \Delta_\alpha \{p_n^{(\alpha)}(x)\} = t(1-t)^{-\alpha-1} F(x, t)$$

$$= \sum_{n=0}^{\infty} t^{n+1} p_n^{(\alpha+1)}(x)$$

which on equating the coefficient of t^n, gives (1.10).

Conversely, we have

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x)$$

or

$$\Delta_\alpha \sum_{n=0}^{\infty} t^n p_n^{(\alpha)}(x) = \sum_{n=1}^{\infty} t^n p_{n-1}^{(\alpha+1)}(x) = t \sum_{n=0}^{\infty} t^n p_n^{(\alpha+1)}(x)$$

$$= tE_\alpha \sum_{n=0}^{\infty} t^n p_n^{(\alpha)}(x)$$

where $\qquad E_\alpha\{f(\alpha)\} = f(\alpha+1).$

Writing $\sum_{n=0}^{\infty} t^n p_n^{(\alpha)}(x) = G$, we get

$$\Delta_\alpha G - tE_\alpha G = 0$$

or

$$[(1 - t) E_\alpha - 1] G = 0.$$

Solving the above homogeneous linear equation, we obtain

$$G = (1 - t)^{-\alpha} F(x, t)$$

where $F(x, t)$ is a periodic function of period unity in α.

This completes the proof of the theorem.

The generating function (2.1) has also been studied by Brown (1969, 1970), Chatterjea (1969) and Mittal (1972, 1974). Starting with (2.1) they have given the generating functions for $\{p_n^{(\alpha+\beta n)}(x)\}$ and $\{p_n^{(-\alpha-mn+1)}(x)\}$. Brown has also studied certain properties of the polynomial sets $\{p_n^{(\alpha)}(x)\}$ and associated sets $\{p_n^{(\alpha+\beta n)}(x)\}$.

It is important to note that in our further analysis here we assume $F(x, t)$ to be independent of α.

Corollary 1 — The set $\{p_n^{(\alpha)}(x)\}$ satisfies the relation (1.10), if and only if there exist polynomial coefficients $a_k(x)$ of degree $\leqslant k$ in x and independent of α, such that

$$p_n^{(\alpha)}(x) = a_0(x) \frac{(\alpha)_n}{n!} + a_1(x) \frac{(\alpha)_{n-1}}{(n-1)!} + \ldots + a_n(x). \qquad \ldots(2.2)$$

The relation (2.2) gives an explicit representation for $p_n^{(\alpha)}(x)$ and shows that $p_n^{(\alpha)}(x)$ is a polynomial of degree n in α also.

From (2.1), we can write the following explicit form for $p_n^{(\alpha)}(x)$ (which can be obtained by putting $\alpha = 0$ in (2.1))

$$p_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(\alpha)_r}{r!} p_{n-r}^{(0)}(x). \qquad \ldots(2.3)$$

Theorem 2 — A necessary and sufficient condition that $p_n^{(\alpha)}(x)$, defined by (1.10), satisfies the recurrence relation

$$np_n^{(\alpha)}(x) = (\alpha + b_0(x))\, p_{n-1}^{(\alpha)}(x) + \ldots + (\alpha + b_{n-1}(x))\, p_0^{(\alpha)}(x), \qquad \ldots(2.4)$$

is that there exists a formal power series

$$B(x, t) = \sum_{n=0}^{\infty} b_n(x)\, t^n \qquad \ldots(2.5)$$

in t, such that

$$B(x, t) = F'(x, t)/F(x, t) \qquad \text{...(2.6)}$$

where prime denotes derivative with respect to t.

PROOF : We have, from Theorem 1

$$(1 - t)^{-\alpha} F(x, t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n.$$

Differentiating both the sides with respect to t, we get

$$\sum_{n=1}^{\infty} n t^{n-1} p_n^{(\alpha)}(x) = [\alpha/(1 - t) + F'(x, t)/F(x, t)](1 - t)^{-\alpha} F(x, t)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} (\alpha + b_r(x))\, t^n p_{n-r}^{(\alpha)}(x),$$

which on equating the coefficients of t^n, gives (2.4).

Conversely, let

$$P_n^{(\alpha)}(x, t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n \quad \text{and} \quad C(x, t) = \sum_{n=0}^{\infty} c_n(x)\, t^n \qquad \text{...(2.7)}$$

where $\qquad r c_r(x) = b_{r-1}(x)$ and $c_0(x) = 0$.

(2.4) is the same as

$$\frac{d}{dt} P^{(\alpha)}(x, t) = \left[\alpha(1 - t)^{-1} + \frac{d}{dt} C(x, t)\right] P^{(\alpha)}(x, t).$$

Integrating, we get

$$\log F(x, t) = C(x, t) + A$$

which on differentiating with respect to t and using (2.7), gives (2.6).

Corollary 2 — A necessary and sufficient condition that a set of polynomials $\{p_n^{(\alpha)}(x)\}$ be of the type (1.10) is that it satisfies the recurrence relation

$$n p_n^{(\alpha)}(x) = (\alpha + b_0(x))\, p_{n-1}^{(\alpha)}(x) + (\alpha + b_1(x))\, p_{n-2}^{(\alpha)}(x)$$

$$+ \dots + (\alpha + b_{n-1}(x))\, p_0^{(\alpha)}(x)$$

where $p_0^{(\alpha)}(x) = 1$ and b's are given by

$$F'(x, t)/F(x, t) = \sum_{n=0}^{\infty} b_n(x)\, t^n. \qquad \text{...(2.8)}$$

PROOF : The first part is obvious from the above Theorem, while the converse can be proved by mathematical induction.

Theorem 3 — A necessary and sufficient condition that the set of polynomials $\{p_n^{(\alpha)}(x)\}$ satisfies the condition (1.10) is that

$$np_n^{(\alpha)}(x) = \sum_{r=1}^{n} \sum_{k=0}^{r-1} \binom{r-1}{k} (-)^{r+k-1}\, b_k(x)\, \Delta_\alpha^r \{p_n^{(\alpha)}(x)\} + \alpha\, \Delta_\alpha \{p_n^{(\alpha)}(x)\} \qquad \ldots(2.9)$$

where $b_k(x)$ are given by (2.8).

PROOF : Differentiating the relation (2.1), with respect to 't', we get

$$\sum_{n=0}^{\infty} nt^n p_n^{(\alpha)}(x) = \alpha \sum_{n=0}^{\infty} t^{n+1} p_n^{(\alpha+1)}(x) + \sum_{n,\, r=0}^{\infty} t^{n+r+1} b_r(x)\, p_n^{(\alpha)}(x)$$

or

$$np_n^{(\alpha)}(x) = \alpha p_{n-1}^{(\alpha+1)}(x) + b_0(x)\, p_{n-1}^{(\alpha)}(x) + \ldots + b_{n-1}(x)\, p_n^{(\alpha)}(x).$$

Since

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x) \Rightarrow p_n^{(\alpha+1)}(x) - p_n^{(\alpha)}(x) = p_{n-1}^{(\alpha+1)}(x),$$

we can write

$$np_n^{(\alpha)}(x) = [(\alpha + b_0(x))\, \Delta_\alpha - (b_0(x) - b_1(x))\, \Delta_\alpha^2 + (b_0(x) - \binom{2}{1} \times b_1(x) + b_2(x))\, \Delta_\alpha^3 - \ldots + (-)^{n-1} (b_0(x) - \binom{n-1}{1} \times b_1(x) + \ldots + (-)^{n-1} b_{n-1}(x))\, \Delta_\alpha^n]\, p_n^{(\alpha)}(x)$$

which is the same as (2.9).

The converse part of the theorem is simple and can be proved by mathematical induction.

The relation (2.4) further suggests the following characterization of the set of polynomials $\{p_n^{(\alpha)}(x)\}$. As in Theorem 2, we have the following result:

Theorem 4 — If $p_n^{(\alpha)}(x)$ be a set of polynomials satisfying the condition (1.10). Then $p_n^{(\alpha)}(x)$ satisfies the equation

$$\sum_r M_r^{(\alpha)}(x)\, p_{n-r}^{(\alpha)}(x) = \lambda_n p_n^{(\alpha)}(x) \qquad \ldots(2.10)$$

where λ_n is given by

$$\lambda_n = nm_1 + n(n-1)\, m_2 + \ldots + [n(n-1) \ldots (n-k+1)]\, m_k \qquad \ldots(2.11)$$

in which $m_1, m_2, \ldots, m_k$ are independent of n.

$M_r^{(\alpha)}(x)$ are defined by

$$\sum_{r=0}^{\infty} M_r^{(\alpha)}(x)\, t^r = m_1 M_{[1]}^{(\alpha)}(x, t) + m_2 M_{[2]}^{(\alpha)}(x, t) + \ldots + m_k M_{[k]}^{(\alpha)}(x, t) \qquad \ldots(2.12)$$

the function $M_{[k]}^{(\alpha)}(x, t)$ are given by

$$M_{[k]}^{(\alpha)}(x, t) = \frac{t^k}{F}\Big[F^{(k)} + \frac{k}{1!}\,\alpha(1-t)^{-1} F^{(k-1)} + \frac{k(k-1)}{2!}\,\alpha(\alpha+1)(1-t)^{-2} F^{(k-2)} + \ldots + (\alpha)_k (1-t)^{-k} F\Big]. \qquad \ldots(2.13)$$

We conclude this section by establishing two more theorems. The polynomials which we now consider not only satisfy the condition (1.10) but are also of Sheffer A-type zero (Sheffer 1939), corresponding to the operator J, where J is defined by

$$J(x, D) \equiv J = \sum_{k=0}^{\infty} c_k D^{k+1}, \;\; c_0 \neq 0 \qquad \ldots(2.14)$$

here c_k's are independent of α and $D \equiv d/dx$.

Theorem 5 — The necessary and sufficient condition that the polynomials $p_n^{(\alpha)}(x)$ satisfy the condition (1.10) and are of Sheffer A-type zero corresponding to the operator, J is that

$$(1-t)^{-\alpha} \exp(xH(t)) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n \qquad \ldots(2.15)$$

where $H(t)$ is independent of α and $J(H(t)) = H(J(t)) = t$.

PROOF : Operating both sides of (2.15) by J and using the well-known result $J(H(t)) = H(J(t)) = t$ (Sheffer 1937), one can easily prove that

$$J\{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha)}(x)$$

i.e. $p_n^{(\alpha)}(x)$ are of Sheffer A-type zero.

Similarly, operating both sides of (2.15) by Δ_α, we can easily show that $p_n^{(\alpha)}(x)$ satisfies (1.10).

This proves the sufficient part of the theorem.

To establish the necessary part, we prove (2.15) while it is given that $p_n^{(\alpha)}(x)$ is of Sheffer A-type zero and satisfies (1.10).

Since $\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x)$, by Theorem 1, we have

$$(1-t)^{-\alpha} F(x, t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n.$$

This gives

$$\sum_{n=0}^{\infty} t^n J\{p_n^{(\alpha)}(x)\} = (1-t)^{-\alpha} \sum_{k=1}^{\infty} c_k D^k \{F(x, t)\}$$

which after some calculations, reduces to

$$\left(\sum_{k=1}^{\infty} c_k D^k - t\right) F = 0. \qquad \ldots(2.16)$$

Solving the above differential equation (2.16), we get

$$F(x, t) = \exp(xH(t)). \qquad \ldots(2.17)$$

This proves the theorem.

Theorem 6 — Let the polynomials $p_n^{(\alpha)}(x)$ be of Sheffer A-type zero corresponding to the operator J and satisfy (1.10). Then $p_n^{(\alpha)}(x)$ satisfies the equation

$$np_n^{(\alpha)}(x) = \sum_{k=0}^{n-1} (\alpha + x(k+1)s_k)\, J^{k+1}\{p_n^{(\alpha)}(x)\} \qquad \ldots(2.18)$$

where J is given by (2.14) and s_k by, $H(t) = \sum_{n=0}^{\infty} s_n t^{n+1}$, $s_0 \neq 0$.

PROOF : The proof is similar to that for Theorem 2.

3. OPERATIONAL REPRESENTATION

Consider the set $\{f_n^{(\alpha)}(x, F)\}$ of polynomials, defined by

$$f_n^{(\alpha)}(x, F) = F(x, \nabla_\alpha)\{(\alpha)_n/n!\} \qquad \ldots(3.1)$$

where $F(x, t)$ is independent of n, given by

$$F(x, t) = \sum_{r=0}^{\infty} d_r(x)\, t^r$$

and $$\nabla_\alpha \{f(\alpha)\} = f(\alpha) - f(\alpha - 1).$$

From (3.1), it follows that

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x, F)\, t^n = F(x, \nabla_\alpha)\,(1 - t)^{-\alpha}.$$

Since, symbolically,

$$\Phi(E_\alpha^{-1})\{b^\alpha\} = b^\alpha \Phi(b^{-1}), \quad \text{with} \quad \Phi(x) = \sum_{r=0}^{\infty} e_r x^r$$

we get

$$(1 - t)^{-\alpha} F(x, t) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x, F)\, t^n.$$

Thus, $f_n^{(\alpha)}(x, F) \equiv p_n^{(\alpha)}(x)$, and we have the following theorem:

Theorem 7 — A necessary and sufficient condition that the polynomials $p_n^{(\alpha)}(x)$ satisfy the condition (1.10), is that it be defined by the operational formula

$$p_n^{(\alpha)}(x) = F(x, \nabla_\alpha)\, \frac{(\alpha)_n}{n!} \qquad \text{...(3.2)}$$

where $F(x, \nabla_\alpha)$ is independent of n and then the polynomial is defined by the generating function

$$(1 - t)^{-\alpha} F(x, t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n.$$

4. A Stieltjes Integral Characterization

Theorem 8 — A polynomial set $\{p_n^{(\alpha)}(x)\}$ satisfies the condition (1.10), if and only if, there exists a function $\psi(x)$ of bounded variation on $(0, \infty)$ with the following properties:

(I) $m_n = \int_0^{\infty} (t)_n\, d\psi(t)$, all exist ...(4.1)

(II) $m_0 \neq 0$.

(III) For $n = 0, 1, 2, \ldots$

$$p_n^{(\alpha)}(x) = \int_0^\infty \frac{(\alpha + t)_n}{n!}\, d\psi(t). \qquad \ldots(4.2)$$

PROOF: If (I) and (II) hold then $p_n^{(\alpha)}(x)$ as given by (4.2) exists for each n and is a polynomial of degree exactly n in α. From (4.2), it is obvious that

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x).$$

Conversely, if $p_n^{(\alpha)}(x)$ satisfies (1.10), and $F(x, t)$ be its determining function: $F(x, t) = \sum_{n=0}^{\infty} p_n^{(0)}(x)\, t^n$. Consider the sequence $\{m_n\}$ such that $m_n = n!\, p_n^{(0)}(x)$.

There exists a function $\psi(t)$ of bounded variation on $(0, \infty)$ [by a theorem of Boas (Widder 1946)] with m_n given by (I) and since $p_0^{(0)}(x) \neq 0$ we have $m_0 \neq 0$. Now denoting the right-hand side of (4.2) by $q_n^{(\alpha)}(x)$,

$$\sum_{n=0}^{\infty} q_n^{(\alpha)}(x)\, u^n = (1 - u)^{-\alpha} \int_0^\infty (1 - u)^{-t}\, d\psi(t).$$

Thus, the determining function for $\{q_n^{(\alpha)}(x)\}$ is

$$F^*(x, u) = \int_0^\infty (1 - u)^{-t}\, d\psi(t) = \sum_{n=0}^{\infty} \int_0^\infty \frac{(t)_n}{n!}\, u^n\, d\psi(t)$$

$$= \sum_{n=0}^{\infty} \frac{m_n u^n}{n!} = \sum_{n=0}^{\infty} p_n^{(0)}(x)\, u^n = F(x, u).$$

Hence $\{q_n^{(\alpha)}(x)\} \equiv \{p_n^{(\alpha)}(x)\}$, so that (4.2) is true.

Corollary 3 — The determining function for $\{p_n^{(\alpha)}(x)\}$ is

$$F(x, u) = \int_0^\infty (1 - u)^{-t}\, d\psi(t) = \sum_{n=0}^{\infty} \frac{m_n u^n}{n!}. \qquad \ldots(4.3)$$

5. Generating Relations

In this section we give a theorem on generating and bilateral generating relations involving $p_n^{(\alpha)}(x)$. For the sake of brevity, only the proof of (5.1) is given. Others can be proved by applying the similar technique. Certain special cases have also been discussed.

Theorem 9 — Let $\{p_n^{(\alpha)}(x)\}$ be a set of polynomials, such that

$$\triangle_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x),$$

then

$$\sum_{n=0}^{\infty} \frac{((c))_n}{((d))_n} p_n^{(\alpha)}(x) \; {}_{C+1}F_D\left[\begin{matrix} -\alpha, (c)+n; t \\ (d)+n; \end{matrix}\right] t^n$$

$$= \sum_{k=0}^{\infty} \frac{((c))_k}{((d))_k} p_k^{(0)}(x)\, t^k \qquad \ldots(5.1)$$

$$\sum_{n=0}^{\infty} \frac{((c))_n}{((d))_n} p_n^{(\alpha)}(x)\, t^n = \sum_{k=0}^{\infty} \frac{((c))_k}{((d))_k} p_k^{(0)}(x) \; {}_{C+1}F_D\left[\begin{matrix} \alpha, (c)+k; t \\ (d)+k; \end{matrix}\right] t^k \qquad \ldots(5.2)$$

and

$$\sum_{n=0}^{\infty} p_n^{(\alpha)}(x) \; {}_{C+1}F_D\left[\begin{matrix} -n, (c); y \\ (d); \end{matrix}\right] t^n = \sum_{n,k=0}^{\infty} \frac{(\alpha)_k}{k!} p_n^{(0)}(x)$$

$$\times \; {}_{C+1}F_D\left[\begin{matrix} -n-k, (c); y \\ (d); \end{matrix}\right] t^{n+k} \qquad \ldots(5.3)$$

subject to suitable conditions on the parameters to ensure the convergence of the series involved.

Proof of (5.1) : By Theorem 1, we have

$$\sum_{n,r=0}^{\infty} \frac{(-\alpha)_r}{r!} p_n^{(\alpha)}(x)\, t^{n+r} = \sum_{k=0}^{\infty} p_k^{(0)}(x)\, t^k.$$

Multiplying both sides by $((c))_\beta/((d))_\beta$ and replacing t by tE_β, (where, $E_\beta(f(\beta)) = f(\beta+1)$), we obtain

$$\sum_{n,r=0}^{\infty} \frac{(-\alpha)_r}{r!} p_n^{(\alpha)}(x)\, t^{n+r} E_\beta^{n+r} \left\{\frac{((c))_\beta}{((d))_\beta}\right\} = \sum_{k=0}^{\infty} p_k^{(0)}(x)\, t^k E_\beta^k \left\{\frac{((c))_\beta}{((d))_\beta}\right\}$$

or

$$\sum_{n,r=0}^{\infty} \frac{(-\alpha)_r}{r!} p_n^{(\alpha)}(x)\, t^{n+r} \frac{((c))_{\beta+n+r}}{((d))_{\beta+n+r}} = \sum_{k=0}^{\infty} p_k^{(0)}(x)\, t^k \frac{((c))_{\beta+k}}{((d))_{\beta+k}}.$$

Now putting $\beta = 0$, and summing the inner series on the left-hand side, we get (5.1).

Special Cases — (I) The substitution $p_n^{(\alpha)}(x) = (2/(x-1))^n\, p_n^{(\beta-n,\alpha)}(x)$ in (5.1), gives

$$\sum_{n=0}^{\infty} \frac{((c))_n\,(2t/(x-1))^n}{((d))_n}\; {}_{C+1}F_D\left[\begin{matrix} -\alpha,\ (c)+n;\ t \\ (d)+n; \end{matrix}\right] p_n^{(\beta-n,\alpha)}(x)$$

$$= F\left[\begin{matrix} 1,\ (c) : -\beta;\ -;\ (1+x)\,t/(1-x),\ t \\ (d) : \quad 1;\ -; \end{matrix}\right], \qquad \ldots(5.4)$$

provided, $C \leqslant D$, $|t| < 1$ and the variable x is so constrained that both sides have a meaning.

(II) In (5.2) by putting, $C = D = 1$, $d_1 = \alpha + 1$, $c_1 = c$ and $p_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)$, we get, the generating relation, due to Rainville [1960, (3), p. 202]:

$$\sum_{n=0}^{\infty} \frac{(c)_n\, L_n^{(\alpha)}(x)}{(\alpha+1)_n}\, t^n = (1-t)^{-c}\; {}_1F_1\left[\begin{matrix} c;\ -xt/(1-t) \\ \alpha+1; \end{matrix}\right]$$

valid for $|t| < 1$ and $\alpha > -1$.

While the substitution $p_n^{(\alpha)}(x) = (\alpha + b)_n$, gives the reduction formula

$$F\left[\begin{matrix} (c) : \alpha;\ b;\ t,\ t \\ (d) : -;\ -; \end{matrix}\right] = {}_{C+1}F_D\left[\begin{matrix} (c);\ \alpha+b;\ t \\ (d); \end{matrix}\right],$$

which has been proved recently, in a different way by Exton (1976).

(III) Lastly, taking $p_n^{(\alpha)}(x) = (\alpha + x)_n/n!$ in (5.3), we obtain the following formula (after some obvious changes)

$$(1-t)^{-\alpha-b}\; {}_{C+1}F_D\left[\begin{matrix} \alpha+b,\ (c);\ -xt/(1-t) \\ (d); \end{matrix}\right] = \sum_{n,r=0}^{\infty} \frac{(\alpha)_n\,(b)_r}{n!\,r!} \times$$

(equation continued on p. 485)

$$\times\ {}_{C+1}F_D\left[\begin{matrix} -n-r,\ (c);\ x \\ (d); \end{matrix}\right] t^{n+r} \qquad \ldots(5.5)$$

where $C \leqslant D$, $|t| < 1$, $|x| < 1$ and $|-xt/(1-t)| < 1$.

6. Algebraic Structure

In this section we briefly study the algebraic structure of the polynomial set $\{p_n^{(\alpha)}(x)\}$.

Consider the set

$$G = \{p_n^{(\alpha)}(x) : \triangle_\alpha\, p_n^{(\alpha)}(x) = p_{n-1}^{(\alpha+1)}(x)\}. \qquad \ldots(6.1)$$

Let $p_n^{(\alpha)}(x)$ and $q_n^{(\alpha)}(x)$ be any two elements of G, on which we define an operation ($*$), in the following manner:

$$p_n^{(\alpha)} * q_n^{(\alpha)} = \sum_{k=0}^{n} p_{n-k}^{(0)}\, q_k^{(\alpha)}. \qquad \ldots(6.2)$$

Here $p_n^{(\alpha)}$ and $q_n^{(\alpha)}$ stands for $p_n^{(\alpha)}(x)$ and $q_n^{(\alpha)}(x)$, respectively.

Theorem 10 — $(G, *)$ forms an Abelian group.

PROOF : Consider

$$\sum_{n=0}^{\infty} (p_n^{(\alpha)} * q_n^{(\alpha)})\, t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} p_{n-k}^{(0)}\, q_k^{(\alpha)} = \sum_{k=0}^{\infty} q_k^{(\alpha)}\, t^k \sum_{n=0}^{\infty} p_n^{(0)}\, t^n$$

$$= (1-t)^{-\alpha}\, F(x, t)\, H(x, t) \qquad \ldots(6.3)$$

where $F(x, t)$ and $H(x, t)$ are the determining functions of $p_n^{(\alpha)}$ and $q_n^{(\alpha)}$, respectively.

Hence, by Theorem 1,

$$\forall,\ p_n^{(\alpha)}(x) \text{ and } q_n^{(\alpha)}(x) \in G \Rightarrow p_n^{(\alpha)}(x) * q_n^{(\alpha)}(x) \in G.$$

The identity element is $I_n^{(\alpha)} = (\alpha)_n/n!$, because

$$(p_n^{(\alpha)} * I_n^{(\alpha)}) = \sum_{k=0}^{n} p_{n-k}^{(0)}\, I_k^{(\alpha)} = \sum_{k=0}^{n} p_{n-k}^{(0)} \frac{(\alpha)_k}{k!} = p_n^{(\alpha)},$$

and similarly

$$(I_n^{(\alpha)} * p_n^{(\alpha)}) = p_n^{(\alpha)}.$$

From (6.3), it is evident that if the determining function of $q_n^{(\alpha)}$ is $(F(x, t))^{-1}$, then

$$p_n^{(\alpha)} * q_n^{(\alpha)} = q_n^{(\alpha)} * p_n^{(\alpha)} = I_n^{(\alpha)}.$$

Thus, if $p_n^{(\alpha)}$ has the determining function $F(x, t)$, then its inverse is the polynomial which has the determining function $(F(x, t))^{-1}$.

The rest of the two properties namely Associative and Commutative can be proved easily.

Hence the theorem is proved.

Next consider the set H whose elements are polynomials $p_n^{(\alpha)}(x)$ satisfying two conditions (I) $\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x)$ and (II) are of Sheffer A-type zero, corresponding to a given operator J. Thus

$$\left.\begin{array}{l} H = \{p_n^{(\alpha)} : \Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x) \\ \text{and} \\ J\{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha)}(x)\}. \end{array}\right\} \quad \ldots(6.4)$$

Obviously,

$$H \subseteq G.$$

Let the operator $\odot$ in H be defined in the following way:

$$p_n^{(\alpha)}(x) \odot q_n^{(\alpha)}(x) = \sum_{k=0}^{n} p_{n-k}^{(0)}(x)\, q_k^{(\alpha+x)}(x); \quad \ldots(6.5)$$

$\forall$, $p_n^{(\alpha)}(x)$ and $q_n^{(\alpha)}(x) \in H$.

Theorem 11 — The set $(H, \odot)$ forms an Abelian group.

PROOF : Proceeding as in Theorem 10, one can show that $(H, \odot)$ is an Abelian Group with $I_n^{(\alpha)}(x) = (\alpha - x)_n/n!$, as its identity element.

Theorem 12 — Let $(K, \bullet)$ be the set given by

$$K = \left\{p_n^{(\alpha)}(x) : \Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x) \text{ and } \frac{d}{dx} p_n^{(\alpha)}(x) = p_{n-1}^{(\alpha)}(x)\right\} \quad \ldots(6.6)$$

and

$$p_n^{(\alpha)}(x) \bullet q_n^{(\alpha)}(x) = \sum_{k=0}^{n} p_{n-k}^{(\alpha)}(0)\, q_k^{(0)}(x);\ \forall\ p_n^{(\alpha)}(x) \text{ and } q_n^{(\alpha)}(x) \in K. \qquad ...(6.7)$$

The set $(K, \bullet)$ is a group, having only one element (i.e. the set K is a singleton set).

Proof : In view of Theorem 6, the generating function for the elements of the set K, can be given by

$$(1 - t)^{-\alpha} \exp(xt) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n. \qquad ...(6.8)$$

Hence the set K has only one element $p_n^{(\alpha)}(x)$ given by (6.8). The other part of the theorem can be proved easily and hence omitted.

Theorem 13 — Let $(G, *)$ be a group given by (6.1) and (6.2). The mapping

$f : G \to G$

such that, $f(p_n^{(\alpha)}) = \Delta_\alpha \{p_n^{(\alpha)}\},\ \forall\ p_n^{(\alpha)} \in G$; is an isomorphism.

Proof : The proof is simple and hence omitted.

7. Some Further Properties of the Polynomials $p_n^{(\alpha)}(x)$

The relation

$$\Delta_\alpha \{p_n^{(\alpha)}(x)\} = p_{n-1}^{(\alpha+1)}(x), \quad n \geqslant 1$$

yields the recurrence relation

$$p_n^{(\alpha+1)}(x) - p_n^{(\alpha)}(x) = p_{n-1}^{(\alpha+1)}(x), \quad n \geqslant 1 \qquad ...(7.1)$$

and by iteration

$$\Delta_\alpha^r \{p_n^{(\alpha)}(x)\} = p_{n-r}^{(\alpha+r)}(x), \quad (n \geqslant r). \qquad ...(7.2)$$

The repeated application of (7.1), gives

$$p_{n-r}^{(\alpha+r)}(x) = \sum_{k=0}^{r} \binom{r}{k} (-)^{r-k}\, p_n^{(\alpha+k)}(x), \quad (n \geqslant r). \qquad ...(7.3)$$

In particular if $n = r$ and $p_0^{(\alpha)}(x) = 1$, (7.3) implies

$$1 = \sum_{k=0}^{r} \binom{r}{k} (-)^{r-k} p_r^{(\alpha+k)}(x). \qquad \ldots(7.4)$$

The substitution $f(\alpha) = p_n^{(\alpha)}(x)$, in the well-known result

$$f(\alpha + \mu) = \sum_{r} \binom{\mu}{r} \Delta_\alpha^r f(\alpha)$$

gives
$$p_n^{(\alpha)}(x) = \sum_{r=0}^{\min(\mu, n)} \binom{\mu}{r} p_{n-r}^{(\alpha+r-\mu)}(x). \qquad \ldots(7.5)$$

Putting $\alpha = 0$ in (7.5), we obtain

$$p_n^{(\mu)}(x) = \sum_{r=0}^{\min(\mu, n)} \binom{\mu}{r} p_{n-r}^{(r)}(x) = \sum_{r=0}^{\min(\mu, n)} \binom{\mu}{r} [\Delta_\alpha^r\, p_n^{(\alpha)}(x)]_{\alpha=0}.$$

Hence, we have

$$p_n^{(\mu)}(x) = \sum_{r=0}^{\min(\mu, n)} \sum_{k=0}^{r} \binom{\mu}{r} \binom{r}{k} (-)^{r-k} p_n^{(k)}(x). \qquad \ldots(7.6)$$

We know that

$$(\Delta_\alpha \alpha \nabla_\alpha)^n = \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} \alpha^{(k)} \nabla_\alpha^k \right\} \nabla_\alpha^n \qquad \ldots(7.7)$$

and

$$(\nabla_\alpha(\alpha + \beta) \Delta_\alpha)^n = \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (\alpha + \beta)_k \Delta_\alpha^k \right\} \nabla_\alpha^n \qquad \ldots(7.8)$$

where $\alpha^{(0)} = 1$, $\alpha^{(k)} = \alpha(\alpha - 1) \ldots (\alpha - k + 1)$ for $k \geqslant 1$.

The above operational formulae give

$$(\Delta_\alpha \alpha \nabla_\alpha)^n p_m^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} \alpha^{(k)} p_{m-n-k}^{(\alpha)}(x) \qquad \ldots(7.9)$$

and $$(\nabla_\alpha(\alpha+\beta)\,\Delta_\alpha)^n\, p_m^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (\alpha+\beta)_k\, p_{m-n-k}^{(\alpha+k)}(x) \quad \ldots(7.10)$$

respectively.

Again, we know that $(m > 0)$

$$D_\alpha^m = \sum_{r=m}^{\infty} \frac{m!}{r!} h_{r,m}\, \Delta_\alpha^r \quad \ldots(7.11)$$

and $$\Delta_\alpha^m = \sum_{r=m}^{\infty} \frac{m!}{r!} k_{r,m} D_\alpha^r \quad \ldots(7.12)$$

in which $h_{r,m}$ and $k_{r,m}$ are Stirling numbers of first and second kind, defined by $h_{n,1} = (-)^{n-1}(n-1)!$, $h_{n,n} = 1$, $h_{n,i} = h_{n-1,i-1} - (n-1)\, h_{n-1,i}$, $k_{n,1} = 1$, $k_{n,n} = 1$ and $k_{n,i} = k_{n-1,i-1} + i k_{n-1,i}$.

From the formulae (7.11) and (7.12), we can write

$$D_\alpha^m\, p_n^{(\alpha)}(x) = \sum_{r=m}^{\infty} \frac{m!}{r!} h_{r,m} p_{n-r}^{(\alpha+r)}(x) \quad \ldots(7.13)$$

and $$p_{n-m}^{(\alpha+m)}(x) = \sum_{r=m}^{\infty} \frac{m!}{r!} k_{r,m} D_\alpha^r\, p_n^{(\alpha)}(x). \quad \ldots(7.14)$$

Substituting the value of $D_\alpha^r\, p_n^{(\alpha)}(x)$ from (7.13) in (7.14), we get

$$p_n^{(\alpha)}(x) = \sum_{r=m}^{\infty} \sum_{s=r}^{\infty} \frac{m!}{s!} h_{s,r} k_{r,m} p_{m+n-s}^{(\alpha+s-m)}(x). \quad \ldots(7.15)$$

If we put $\alpha = 0$, in (2.1), we get

$$F(x, t) = \sum_{n=0}^{\infty} p_n^{(0)}(x)\, t^n. \quad \ldots(7.16)$$

Hence (2.1) together with (7.6), gives the expansion

$$p_n^{(0)}(x) = \sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} p_{n-k}^{(\alpha)}(x). \quad \ldots(7.17)$$

By putting the value of $p_n^{(0)}(x)$ from (7.17) in (2.3), one can show that

$$p_n^{(\alpha)}(x) = \sum_{r=0}^{n} \sum_{k=0}^{r} \frac{(\alpha)_{n-r}\,(-\alpha)_k}{(n-r)!\,k!}\, p_{r-k}^{(\alpha)}(x). \qquad \text{...(7.18)}$$

From (2.1), we have

$$\sum_{n=0}^{\infty} p_n^{(\alpha+\beta)}(x)\, t^n = (1-t)^{-\alpha}\,[(1-t)^{-\beta} F(x,t)]$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!}\, t^r \sum_{n=0}^{\infty} p_n^{(\beta)}(x)\, t^n,$$

on equating the coefficient of t^n and replacing α by $\alpha - \beta$ we get

$$p_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(\alpha-\beta)_{n-r}}{(n-r)!}\, p_r^{(\beta)}(x). \qquad \text{...(7.19)}$$

The substitution $\beta = \alpha - 1$, in (7.19), gives the interesting result

$$p_n^{(\alpha)}(x) = \sum_{r=0}^{n} p_r^{(\alpha-1)}(x). \qquad \text{...(7.20)}$$

8. Conclusion

We conclude this paper with the remark that the polynomial sets $\{p_n^{(\alpha)}(x)\}$ can be generalized further by considering the linear difference operator of infinite order

$$\tau \equiv \tau(\Delta_\alpha) = \sum_{k=1}^{\infty} h_k^{(\alpha)}\, \Delta_\alpha^k\,,\; h_1^{(\alpha)} \neq 0$$

as a generalization of the finite difference operator Δ_α. In this case we take the polynomial sets $\{f_n^{(\alpha)}(x)\}$ which satisfy the functional relation

$$\tau\,\{f_n^{(\alpha)}(x)\} = f_{n-1}^{(\alpha+1)}(x), \quad (n = 1, 2, \ldots).$$

The polynomials $\{f_n^{(\alpha)}(x)\}$ can be further generalized in the following form:

Consider the polynomial sets $\{f_n^{(\alpha)}(x)\}$ which have the property

$$\tau\,\{f_n^{(\alpha)}(x)\} = f_{n-r}^{(\alpha+s)}(x), \quad (n = r, r+1, r+2, \ldots)$$

where r and s are fixed positive integers. This time τ is given by

$$\tau = \sum_{k=0}^{\infty} g_k^{(\alpha)} \Delta_{\alpha}^{k+r}, g_0^{(\alpha)} \neq 0.$$

We shall study the above two generalizations in our subsequent communications.

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