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CONSTRAINED EXTREMAL PROBLEMS FOR SYMMETRIC FUNCTIONS WITH REAL PART POSITIVE

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In the unit disc $E = \{z \mid |z| < 1\}$. Let P denote the class of regular analytic functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ with Re $p(z) > 0$ and $p_1, p_2, ...$ are all real. Let S^*, T and I denote, respectively, the classes of symmetric starlike functions, typically real functions and functions which map E onto a domain convex in the direction of imaginary axis. Jenkins (1961) obtained distortion theorems for $f(z)$ and $f'(z) \in T$ and $f''(0)$ fixed when z is real by using an application of Neyman-Pearson Lemma. In this paper we have developed function theoretic technique to give complete solution of the problem of finding the extreme values of Re $F(p(c))$, $p \in P$, when $p'(0)$ is fixed, where F is an analytic function in the right half plane. This is then applied to obtain distortion theorems for the classes S^*, T and I when the second derivative has assigned value at the origin.

Let P be the class of analytic functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in $E = \{z \mid |z| < 1\}$ with Re $p(z) > 0$ there such that the coefficients $p_1, p_2, ...$ are all real and let P_1 denote that sub-class of P whose functions have fixed second coefficient $p_1 = 2a_1$, $-1 \leqslant a_1 \leqslant 1$. Let S^*, T and I denote, the classes of regular analytic functions $f(z)$ in E, $f(0) = f'(0) - 1 = 0$, which are respectively, related to functions of the class P by the equations,

$$\frac{zf'(z)}{f(z)} = p(z) \qquad ...(1)$$

$$f(z) = \frac{z}{1 - z^2} p(z) \qquad ...(2)$$

and

$$f(z) = \int_0^z \frac{p(z)}{1 - z^2} dz. \qquad ...(3)$$

The class S^* is clearly the class of symmetric starlike functions. The class T was introduced by Rogosinski (1932) and is known as the class of typically real functions and the class I, introduced by Robertson (1936), is known as the class of functions which map E onto a domain convex in the direction of the imaginary axis.

The class I is the subclass of the class S of univalent functions and the class T contains the class S_R of symmetric univalent functions and possesses properties which are similar to those of S_R. By S_1^*, T_1 and I_1 we shall denote the subclasses of S^*, T and I respectively, whose functions have fixed second coefficient. The extremal problems for the classes S_1^*, T_1 and I_1 lead to extremal problem for the class P_1. The elements of the class P can be represented in terms of Stieltjes integral (Robertson 1935) analogous to the well-known Herglotz formula (1911) and this fact has been used to characterize the form of extremal function for constrained extremal problems for the class P_1. A typical result of this type is the following due to Leeman (1973).

Theorem A — Let G be real-valued and continuous on $[0, \pi]$. Then the functional Φ defined on the set of all functions of the form

$$f(z) = \int_0^\pi \frac{z}{1 - 2z\cos\theta + z^2}\, d\alpha(\theta) \text{ by } \Phi(f) = \int_0^\pi G(\theta)\, d\alpha(\theta)$$

assumes its minimum and maximum value in T_1 for a function of the form

$$f(z) = \frac{2a_1 - t}{s - t}\,\frac{z}{1 - sz + z^2} + \frac{s - 2a_1}{s - t}\,\frac{z}{1 - tz + z^2}, \qquad \ldots(4)$$

$$-2 \leqslant s \leqslant 2a_1 \leqslant t \leqslant 2;$$

where for $s = t = 2a_1$ we define $f(z) = \dfrac{z}{1 - 2a_1z + z^2}$.

This theorem reduces the solution of a constrained extremal problem to that of a problem in classical differential calculus. Jenkins (1961) had used the Neyman-Pearson Lemma to obtain distortion theorems for $f(z)$ and $f'(z)$, $f(z) \in T_1$, when z is real. This idea of Jenkins had been extended by Lai (1963) to the class of starlike typically real functions.

In the present paper we give a method which helps in solving extremal problems in the class P_1. We first express any function of P_1 in terms of a function of the class P and thus get rid of the constraint that the second coefficient is fixed. We then use the adaptation by Gupta (1973) of the variational formula of Sakaguchi (1964) for the class P to obtain complete solution. Further we also obtain a direct function theoretic proof of the results of Jenkins (1961).

Theorem 1 — Let $f \in T_1$ and have the expansion about the origin

$$f(z) = z + 2a_1z^2 + \ldots, \quad -1 \leqslant a_1 \leqslant 1.$$

For a fixed value of a_1, $0 < r < 1$,

$$\frac{r}{1 - 2a_1r + r^2} \leqslant f(r) \leqslant \frac{r(1 + 2a_1r + r^2)}{(1 - r^2)^2} \qquad \ldots(5)$$

$$\frac{1-r^2}{(1-2a_1r+r^2)^2} \leqslant f'(r) \leqslant \tfrac{1}{2}\left[(1+a_1)\frac{(1+r)}{(1-r)^3} + (1-a_1)\frac{(1-r)}{(1+r)^3}\right] \qquad \text{...(6)}$$

$$\frac{1-r^2}{1-2a_1r+r^2} \leqslant r\,\frac{f'(r)}{f(r)} \leqslant \frac{(1+r^2)^2+4r^2+4a_1r(1+r^2)}{(1-r^2)(1+2a_1r+r^2)}. \qquad \text{...(7)}$$

Left equalities are attained only for the function $f(z) = \dfrac{z}{1-2a_1z+z^2}$ and right by $f(z) = \dfrac{z(1+2a_1z+z^2)}{(1-z^2)^2}$.

PROOF: Let $p \in P$ be arbitrary, then it is easily seen (Singh and Paul 1977) that the function

$$p_1(z) = \frac{(1-a_1)(1-z)+(1+a_1)(1+z)\,p(z)}{(1-a_1)(1+z)+(1+a_1)(1-z)\,p(z)}, \qquad \text{...(8)}$$

$$-1 \leqslant a_1 \leqslant 1, \quad z \in E$$

belongs to P_1.

Now consider the function

$$\phi(z) = \frac{p_1(z)-p_1(r)}{p_1(z)+p_1(r)}\,\frac{1-rz}{z-r}, \quad r \text{ real}. \qquad \text{...(9)}$$

$\phi(z)$ is symmetric, $|\phi(z)| < 1$ and

$$\phi(0) = -\frac{1-p_1(r)}{1+p_1(r)}\,\frac{1}{r} \text{ is real}. \qquad \text{...(10)}$$

The function

$$\psi(z) = \frac{\phi(z)-\phi(0)}{1-\phi(0)\,\phi(z)},\ \psi(0) = 0 \qquad \text{...(11)}$$

is also symmetric and $|\psi(z)| \leqslant |z|$.

Since

$$\lim_{z\to r} \phi(z) = \frac{p_1'(r)}{2p_1(r)}(1-r^2) \qquad \text{...(12)}$$

and $\phi(r)$ is a monotone function of $\psi(r)$ and $\psi(r)$ satisfies $-r \leqslant \psi(r) \leqslant r$, we obtain

$$\frac{\phi(0)-r}{1-r\,\phi(0)} \leqslant \frac{p_1'(r)}{2p_1(r)}(1-r^2) \leqslant \frac{r+\phi(0)}{1+r\phi(0)}. \qquad \text{...(13)}$$

Substituting the value of $\phi(0)$ from (10), we have

$$p_1(r) - \frac{1+r^2}{1-r^2} \leqslant r\,\frac{p_1'(r)}{p_1(r)} \leqslant \frac{1+r^2}{1-r^2} - \frac{1}{p_1(r)}. \qquad \text{...(14)}$$

From (8) it readily follows that $p_1(r)$ is a monotone increasing function of $p(r)$ and because $\frac{1-r}{1+r} \leqslant p(r) \leqslant \frac{1+r}{1-r}$, we get

$$\frac{1-r^2}{1-2a_1r+r^2} \leqslant p_1(r) \leqslant \frac{1+2a_1r+r^2}{1-r^2}. \qquad \ldots(15)$$

In (14) we need maximum of right side and minimum of left side, that is

$$\min p_1(r) - \frac{1+r^2}{1-r^2} \leqslant r\,\frac{p_1'(r)}{p_1(r)} \leqslant \frac{1+r^2}{1-r^2} - \frac{1}{\max p_1(r)}.$$

Thus we have

$$-\frac{2r\{2r-a_1(1+r^2)\}}{(1-r^2)(1-2a_1r+r^2)} \leqslant r\,\frac{p_1'(r)}{p_1(r)} \leqslant \frac{2r\{2r+a_1(1+r^2)\}}{(1-r^2)(1+2a_1r+r^2)}. \qquad \ldots(16)$$

(15) and (16) along with the relation (2) yield (5) and (7) respectively and (6) follows from

$$\frac{f(r)}{r}\,\frac{1-r^2}{1-2a_1r+r^2} \leqslant f'(r) \leqslant \frac{f(r)}{r}\,\frac{(1+r^2)^2+4r^2+4a_1r(1+r^2)}{(1-r^2)(1+2a_1r+r^2)} \qquad \ldots(17)$$

on using the extreme values of $f(r)$ given by (5).

Corollary 1 — If $g \in I_1$ and have the expansion about the origin

$$g(z) = z + a_1z^2 + \ldots, \quad -1 \leqslant a_1 \leqslant 1.$$

For a fixed value of a_1, $0 < r < 1$,

$$\frac{1}{(1-a_1^2)^{1/2}} \tan^{-1} \frac{(1-a_1^2)^{1/2}\,r}{1-a_1r} \leqslant g(r) \leqslant \frac{r(1+a_1r)}{1-r^2} \qquad \ldots(18)$$

$$\frac{1-r^2}{(1+a_1r)(1-2a_1r+r^2)} \leqslant r\,\frac{g'(r)}{g(r)}$$

$$\leqslant \frac{r(1+2a_1r+r^2)(1-a_1^2)^{1/2}}{(1-r^2)^2\tan^{-1}\{(1-a_1^2)^{1/2}\,r/(1-a_1r)\}} \qquad \ldots(19)$$

$$\frac{2r(a_1-r)}{1-2a_1r+r^2} \leqslant r\,\frac{g''(r)}{g'(r)} \leqslant \frac{2r[3r(1+a_1r)+(r^3+a_1)]}{(1-r^2)(1+2a_1r+r^2)}. \qquad \ldots(20)$$

(18) and (20) are sharp.

Proof follows from Theorem 1 by using the relation $f(z) = zg'(z)$, $f(z) \in T_1$.

We shall need the following variational formula (Gupta 1973, Ch. II, p. 80) for the class P.

Theorem 2 — Let $p \in P$ then there exists a function $p^*(z) = p(z) + \delta p(z)$ belonging to P and with $\delta p(z)$ of the form

$$\frac{2}{\rho}\,\delta p(z) = \epsilon\left[p(z)\left(\frac{1}{1-\bar{\alpha} z} + \frac{z}{z-\bar{\alpha}} - 1\right) + \overline{p(\alpha)}\left(\frac{\bar{\alpha} z}{1-\bar{\alpha} z} + \frac{\bar{\alpha}}{\bar{\alpha}-z} - p(z)\right)\right]$$
$$+ \bar{\epsilon}\left[p(z)\left(\frac{1}{1-\alpha z} + \frac{z}{z-\alpha} - 1\right) + p(\alpha)\left(\frac{\alpha z}{1-\alpha z} + \frac{\alpha}{\alpha - z} - p(z)\right)\right]$$
$$+ o(1) \qquad \ldots(21)$$

where α and ϵ are arbitrary complex numbers such that $|\alpha| < 1$, $|\epsilon| = 1$ and ρ is sufficiently small positive number.

We shall now prove the following for symmetric functions. We may assume $\operatorname{Im} z \geqslant 0$ because the other case can be obtained by reflection.

Theorem 3 — Let $c \in E$, $c \neq 0$, $\operatorname{Im} c \geqslant 0$ be fixed and let $F(w)$ be analytic in $\operatorname{Re} w > 0$. If $p_1 \in P_1$ gives the extreme value of $\operatorname{Re} F(p_1(c))$, $\lambda = F_w(p_1(c)) = |\lambda| e^{i\alpha} \neq 0$, $0 \leqslant \alpha \leqslant \pi/2$, then $p_1(z)$ has either the form

$$p_1(z) = \frac{1+z}{1-z}\,\frac{1+z^2-(1-a_1)z-(1+a_1)zt}{1+z^2+(1-a_1)z-(1+a_1)zt}, \quad -1 \leqslant t \leqslant 1; \quad \ldots(22)$$

or

$$p_1(z) = \frac{1-z}{1+z}\,\frac{1+z^2+(1+a_1)z-(1-a_1)zt}{1+z^2-(1+a_1)z-(1-a_1)zt}, \quad -1 \leqslant t \leqslant 1. \quad \ldots(23)$$

The following cases arise :

Case I — The function defined by (22) gives the minimum $\operatorname{Re} F(p_1(c))$ where

$$t = t_1 = \frac{\operatorname{Re}\left\{\lambda\dfrac{1+c}{1-c}\left(\dfrac{1+\bar{c}^2}{\bar{c}} + 1 - a_1\right)\right\} - |\lambda|\left|\dfrac{1+c}{1-c}\right|\dfrac{1-|c|^2}{|c|^2}\operatorname{Im} c}{(1+a_1)\operatorname{Re}\left(\lambda\dfrac{1+c}{1-c}\right)}$$

if

$$\frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\,|1+c|^2} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}\operatorname{Re}\left(\lambda\frac{1+c}{1-c}\right)$$
$$\geqslant \operatorname{Re}\left(\lambda\frac{c}{1-c^2}\right) \geqslant \frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\,|1+c|^2} \qquad \ldots(24)$$

and it gives the maximum where

$$t = t_2 = \frac{\operatorname{Re}\left\{\lambda\dfrac{1+c}{1-c}\left(\dfrac{1+\bar{c}^2}{\bar{c}} + 1 - a_1\right)\right\} + |\lambda|\left|\dfrac{1+c}{1-c}\right|\dfrac{1-|c|^2}{|c|^2}\operatorname{Im} c}{(1+a_1)\operatorname{Re}\left(\lambda\dfrac{1+c}{1-c}\right)}$$

if $\operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right) \neq 0$ and

$$-\frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1+c|^2 \operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right)} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}$$

$$\geqslant \frac{\operatorname{Re}\left(\lambda \frac{c}{1-c^2}\right)}{\operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right)} \geqslant -\frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1+c|^2 \operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right)}. \quad \ldots(25)$$

If $\operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right) = 0$, the maximum is attained for

$$t = t_3 = \frac{(1+|c|^2)\operatorname{Re} c + (1-a_1)|c|^2}{(1+a_1)|c|^2}$$

where $t_3 = \lim_{\operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right) \to 0} t_2.$

Case II — The function defined by (23) gives the minimum $\operatorname{Re} F(p_1(c))$ where

$$t = t_4 = \frac{\operatorname{Re}\left\{\lambda \frac{1-c}{1+c}\left(\frac{1+\bar{c}^2}{\bar{c}} - 1 - a_1\right)\right\} + |\lambda| \left|\frac{1-c}{1+c}\right| \frac{1-|c|^2}{|c|^2} \operatorname{Im} c}{(1-a_1)\operatorname{Re}\left(\lambda \frac{1-c}{1+c}\right)}$$

if

$$-\frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1-c|^2} \geqslant \operatorname{Re}\left(\lambda \frac{c}{1-c^2}\right)$$

$$\geqslant -\frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1-c|^2} - 2(1-a_1)\frac{|c|^2}{|1-c|^4}\operatorname{Re}\left(\lambda \frac{1-c}{1+c}\right) \quad \ldots(26)$$

and it gives the maximum where

$$t = t_5 = \frac{\operatorname{Re}\left\{\lambda \frac{1-c}{1+c}\left(\frac{1+\bar{c}^2}{\bar{c}} - 1 - a_1\right)\right\} - |\lambda| \left|\frac{1-c}{1+c}\right| \frac{1-|c|^2}{|c|^2} \operatorname{Im} c}{(1-a_1)\operatorname{Re}\left(\lambda \frac{1-c}{1+c}\right)}$$

if

$$\frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1-c|^2} \geqslant \operatorname{Re}\left(\lambda \frac{c}{1-c^2}\right) \geqslant \frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1-c|^2} -$$

(equation continued on p. 498)

$$- 2(1 - a_1) \frac{|c|^2}{|1 - c|^4} \operatorname{Re}\left(\lambda \frac{1 - c}{1 + c}\right). \quad \ldots(27)$$

Case III — $p_1(z) = \dfrac{1 + 2a_1 z + z^2}{1 - z^2}$, gives the minimum $\operatorname{Re} F(p_1(c))$ if

$$- \frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 - c|^2} \leqslant \operatorname{Re}\left(\lambda \frac{c}{1 - c^2}\right) \leqslant \frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 + c|^2} \quad \ldots(28)$$

and it gives the maximum either if

$$\frac{\operatorname{Re}\left(\lambda \dfrac{c}{1 - c^2}\right)}{\operatorname{Re}\left(\lambda \dfrac{1 + c}{1 - c}\right)} \leqslant - \frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 + c|^2 \operatorname{Re}\left(\lambda \dfrac{1 + c}{1 - c}\right)},$$

$$\operatorname{Re}\left(\lambda \frac{1 + c}{1 - c}\right) \neq 0 \quad \ldots(29)$$

or if

$$\frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 - c|^2} \leqslant \operatorname{Re}\left(\lambda \frac{c}{1 - c^2}\right). \quad \ldots(30)$$

Case IV — $p_1(z) = \dfrac{1 - z^2}{1 - 2a_1 z + z^2}$ gives the minimum $\operatorname{Re} F(p_1(c))$ either if

$$\operatorname{Re}\left(\lambda \frac{c}{1 - c^2}\right) \geqslant \frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 + c|^2} + 2(1 + a_1) \frac{|c|^2}{|1 + c|^4} \operatorname{Re}\left(\lambda \frac{1 + c}{1 - c}\right) \quad \ldots(31)$$

or if

$$\operatorname{Re}\left(\lambda \frac{c}{1 - c^2}\right) \leqslant - \frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 - c|^2} - 2(1 - a_1) \frac{|c|^2}{|1 - c|^4} \operatorname{Re}\left(\lambda \frac{1 - c}{1 + c}\right) \quad \ldots(32)$$

and it gives the maximum either if

$$- \frac{|\lambda|(1 - |c|^2)\operatorname{Im} c}{|1 - c^2|\ |1 + c|^2 \operatorname{Re}\left(\lambda \dfrac{1 + c}{1 - c}\right)} + 2(1 + a_1) \frac{|c|^2}{|1 + c|^4} \leqslant \frac{\operatorname{Re}\left(\lambda \dfrac{c}{1 - c^2}\right)}{\operatorname{Re}\left(\lambda \dfrac{1 + c}{1 - c}\right)} \quad \ldots(33)$$

or if

$$\operatorname{Re}\left(\lambda \frac{c}{1-c^2}\right) \leqslant \frac{|\lambda|(1-|c|^2)\operatorname{Im} c}{|1-c^2|\;|1-c|^2} - 2(1-a_1)\frac{|c|^2}{|1-c|^4}\operatorname{Re}\left(\lambda\frac{1-c}{1+c}\right). \quad \ldots(34)$$

Since P_1 is compact, there exists $p_1 \in P_1$ which minimizes or maximizes $\operatorname{Re} F(p_1(c))$. Let δ denote the variation, we then have

$$\delta \operatorname{Re} F(p_1(c)) = \operatorname{Re}\{\lambda \delta p_1(c)\} + o(1),\; \lambda = F_w(p_1(c)). \quad \ldots(35)$$

Now let $p \in P$ be arbitrary then the function

$$p_1(z) = \frac{(1-a_1)(1-z)+(1+a_1)(1+z)p(z)}{(1-a_1)(1+z)+(1+a_1)(1-z)p(z)},\; -1 \leqslant a_1 \leqslant 1,\, z \in E$$

belongs to P_1 and

$$\delta p_1(c) = \frac{4c(1-a_1^2)\,\delta p(c)}{[(1-a_1)(1+c)+(1+a_1)(1-c)p(c)]^2} + o(1),\; p \in P. \quad \ldots(36)$$

Using the expression for $\delta p(c)$ from Theorem 2, we get

$$\delta \operatorname{Re} F(p_1(c)) = \frac{\rho}{2}\operatorname{Re}\bar{\epsilon}\Big[\bar{\lambda}\bar{X}\overline{p(c)}\left(\frac{1}{1-\alpha\bar{c}}+\frac{\bar{c}}{\bar{c}-\alpha}-1\right) + \lambda X p(c)\left(\frac{1}{1-\alpha c}+\frac{c}{c-\alpha}-1\right) + p(\alpha)\Big\{\bar{\lambda}\bar{X}\left(\frac{\alpha\bar{c}}{1-\alpha\bar{c}} + \frac{\alpha}{\alpha-\bar{c}} - \overline{p(c)}\right) + \lambda X\left(\frac{\alpha c}{1-\alpha c}+\frac{\alpha}{\alpha-c}-p(c)\right)\Big\}\Big] + o(1) \quad \ldots(37)$$

where

$$X = \frac{4(1-a_1^2)\,c}{[(1-a_1)(1+c)+(1+a_1)(1-c)p(c)]^2} \quad \ldots(38)$$

$|\epsilon| = 1$, $\alpha \in E$ and $\rho > 0$ is arbitrary small.

The usual variational argument then gives that the expression with in the square brackets in the right-hand side of (37) must vanish. Since in this expression $\alpha \in E$ is arbitrary, replacing it by z, we obtain

$$p(z) = A(z)/B(z) \quad \ldots(39)$$

where

$$A(z) = -\left[\bar{\lambda}\bar{X}\overline{p(c)}\left(\frac{1}{1-\bar{c}z}+\frac{\bar{c}}{\bar{c}-z}-1\right)+\right.$$

(*equation continued on p. 500*)

$$+ \lambda X p(c)\left(\frac{1}{1-cz}+\frac{c}{c-z}-1\right)\Bigg]$$

$$= -\frac{2\,|\,c\,|^{2}\,\mathrm{Re}\,(\lambda Xp)\,(1-z^{2})\left[1 - z\,\dfrac{\mathrm{Re}\,\{\lambda Xp(1+\bar{c}^{2})\,c\}}{|\,c\,|^{2}\,\mathrm{Re}\,(\lambda Xp)} + z^{2}\right]}{(1-\bar{c}z)\,(\bar{c}-z)\,(1-cz)\,(c-z)} \qquad \ldots(40)$$

$$B(z) = \bar{\lambda}\bar{X}\left(\frac{z\bar{c}}{1-z\bar{c}}+\frac{z}{z-\bar{c}}-\overline{p(c)}\right)+\lambda X\left(\frac{zc}{1-zc}+\frac{z}{z-c}-p(c)\right)$$

$$= \frac{-2\,|\,c\,|^{2}\,\mathrm{Re}\,(\lambda Xp)\,[1 + A_1 z + A_2 z^{2} + A_1 z^{3} + z^{4}]}{(1-\bar{c}z)\,(\bar{c}-z)\,(1-cz)\,(c-z)} \qquad \ldots(41)$$

and

$$\begin{cases} A_1 = -2\,\mathrm{Re}\left[\dfrac{1+c^{2}}{c}\right] + \dfrac{\mathrm{Re}\,\{\lambda X(1-c^{2})\,\bar{c}\}}{|\,c\,|^{2}\,\mathrm{Re}\,(\lambda Xp)} \\ A_2 = 2 + \left|\dfrac{1+c^{2}}{c}\right|^{2} + \dfrac{\mathrm{Re}\,\{\lambda X(1+\bar{c}^{2})\,(1-c^{2})\}}{|\,c\,|^{2}\,\mathrm{Re}\,(\lambda Xp)} \end{cases} \qquad \ldots(42)$$

provided $B(z) \not\equiv 0$ and $\mathrm{Re}\,(\lambda Xp) \neq 0$. However $B(z) \equiv 0$ only if $\lambda = 0$. If $\mathrm{Re}\,(\lambda Xp) = 0$, we have

$$p(z) = \frac{(1-z^{2})\,(1-|\,c\,|^{2})\,\mathrm{Im}\,c}{(1+z^{2})\,\mathrm{Im}\,(c(1+\bar{c}^{2})/p) + z\{(1+\bar{c}^{2})\,(1-c^{2})/p\}}. \qquad \ldots(43)$$

For the proof of Theorem 4 we need the lemmas given below.

Lemma 1 — If $p_1(z)$ gives minimum $\mathrm{Re}\,F(p_1(c))$ then $B(z) \geqslant 0$ on $|\,z\,| = 1$ and if $p_1(z)$ gives maximum $\mathrm{Re}\,F(p_1(c))$, then $B(z) \leqslant 0$ on $|\,z\,| = 1$.

PROOF : We shall prove the case of minimum for the other case is exactly similar and can easily be derived. We note that the extremal function $p_1(z) \in P_1$ will be of the form given by (8) for some $p(z) \in P$. Let $p(z) \in P$ in (8) correspond to the extremal function $p_1(z)$. Then

$$p^{*}(z) = \frac{p(z) + \rho\,\frac{1}{2}\left(\dfrac{1+\bar{\alpha}z}{1-\bar{\alpha}z}+\dfrac{1+\alpha z}{1-\alpha z}\right)}{1+\rho},\quad \rho > 0,\ \alpha \in E \qquad \ldots(44)$$

also belongs to P and $p_1^{*}(z)$ given by (8) for which $p(z)$ is replaced by $p^{*}(z)$ given by (44) belongs to P_1. It is easily verified that

$$\delta p_1(c) = p_1^{*}(c) - p_1(c) =$$

$$\frac{4c(1-a_1^{2})\,\rho\left[\frac{1}{2}\left(\dfrac{1+\bar{\alpha}c}{1-\bar{\alpha}c}+\dfrac{1+\alpha c}{1-\alpha c}\right)-p(c)\right]}{[(1-a_1)\,(1+c)+(1+a_1)\,(1-c)\,p(c)]^{2}} + o\,(\rho^{2}). \qquad \ldots(45)$$

For this variation in view of (35), we get

$$\delta \operatorname{Re} F(p_1(c)) = \rho \operatorname{Re}\left[\lambda X\left\{\tfrac{1}{2}\left(\frac{1+\bar{\alpha}c}{1-\bar{\alpha}c}+\frac{1+\alpha c}{1-\alpha c}\right)-p(c)\right\}\right]+o(\rho^2). \quad \ldots(46)$$

Since for minimum Re $F(p_1(c))$, δ Re $F(p_1(c) \geqslant 0$, on dividing by $\rho > 0$ and taking limit $\rho \to 0$ we obtain

$$\operatorname{Re}\left[\lambda X\left\{\tfrac{1}{2}\left(\frac{1+\bar{\alpha}c}{1-\bar{\alpha}c}+\frac{1+\alpha c}{1-\alpha c}\right)-p(c)\right\}\right] \geqslant 0. \quad \ldots(47)$$

As $\alpha \in E$ is arbitrary, letting $|\alpha| \to 1$ and putting $\alpha = z$, we have

$$B(z) = 2 \operatorname{Re}\left[\lambda X\left(\frac{z}{z-c}+\frac{zc}{1-zc}-p(c)\right)\right] \geqslant 0 \text{ on } |z| = 1.$$

Lemma 2 — The extremal function $p_1 \in P_1$ has either of the forms given by (22) or (23)

PROOF : We note that the form of extremal function $p_1 \in P_1$ is given by (8) where $p(z)$ is given by (39) if Re $(\lambda X p) \neq 0$ and by (43) if Re $(\lambda X p) = 0$.

Let Re $(\lambda X p) \neq 0$, $p(z)$ is given by (39) where $A(z)$ and $B(z)$ are given by (40) and (41). From this we see that $p(z)$ is a rational function of z with Re $p(z) = 0$ on $|z| = 1$. It must have at least one pole on $|z| = 1$ and all its poles on $|z| = 1$ must be simple (Sakaguchi 1964) and symmetric with respect to the real axis. It follows that $B(z)$ must have atleast one zero on $|z| = 1$ because $A(z)$ has no poles there. From Lemma 1 we see that $B(z)$ keeps the same sign on $|z| = 1$. Consequently the zeros of $B(z)$ on $|z| = 1$ must be of even order. From (41) the numerator of $B(z)$ is a polynomial of degree four. Hence $B(z)$ can have atmost two zeros each multiple zero being counted once. Since $p(z)$ is symmetric with respect to the real axis and Re $p(z) \geqslant 0$, the standard argument shows that either

$$p(z) = \frac{1-2tz+z^2}{1-z^2}, \quad -1 \leqslant t \leqslant 1 \quad \ldots(48)$$

or

$$p(z) = \frac{1-z^2}{1-2tz+z^2}, \quad -1 \leqslant t \leqslant 1. \quad \ldots(49)$$

However, if Re $(\lambda X p) = 0$, by using the same argument that the zero of denominator of $p(z)$ given by (43) is of even order, it is easy to see that $p(z)$ is either $\frac{1+z}{1-z}$ or $\frac{1-z}{1+z}$.

Further in the case of Im $c = 0$, it is easy to verify from (39) – (41) that $p(z)$ is either $\frac{1+z}{1-z}$ or $\frac{1-z}{1+z}$. From (8) by substituting the values of $p(z)$ given by (48) and (49) we obtain (22) and (23) respectively.

Lemma 3 — If X is given by (38) then minimum Re $F(p_1(c))$ is given by (48) if Re $(\lambda Xp) \geqslant 0$ and by (49) if Re $(\lambda Xp) \leqslant 0$.

For maximum Re $F(p_1(c))$, (48) is the extremal function for Re $(\lambda Xp) \leqslant 0$ and (49) is the extremal function for Re $(\lambda Xp) \geqslant 0$.

PROOF : We shall prove the case of minimum only as the other case is exactly similar.

From Lemma 2, the corresponding extremal function $p(z)$ is either of the form (48) or (49). We show that the extremal function giving minimum for Re $(\lambda Xp) \geqslant 0$ cannot be of the form (49) and for Re $(\lambda Xp) \leqslant 0$ it cannot be of the form (48).

Let Re $(\lambda Xp) \geqslant 0$, indeed for function of the form (49)

$$[1 + A_1 z + A_2 z^2 + A_1 z^3 + z^4] = (1 - 2tz + z^2)^2 \qquad \ldots(50)$$

where A_1, A_2 are given by (42). Comparing the coefficients of different powers of z on both sides of (50), we obtain

$$A_1 = -4t, \; A_2 = 4t^2 + 2. \qquad \ldots(51)$$

From Lemma 1, $p(z) = \frac{1 - z^2}{1 - 2tz + z^2}$, $-1 \leqslant t \leqslant 1$ will give minimum Re $F(p_1(c))$ on $|z| = 1$ iff

$$\left[\frac{1}{z^2} + \frac{A_1}{z} + A_2 + A_1 z + z^2\right] \leqslant 0 \text{ on } |z| = 1,$$

which is not possible because of (50).

Similarly we can prove that for Re $(\lambda Xp) \leqslant 0$ the extremal function giving minimum cannot be of the form (48).

Proof of case I — The extremal function $p_1(z)$ defined by (22) corresponds to the function $p(z) = \frac{1 - 2tz + z^2}{1 - z^2}$, $-1 \leqslant t \leqslant 1$. From (39) – (41) we may write

$$p(z) = \frac{(1 - z^2)\left(1 - z\,\frac{\text{Re}\,\{\lambda Xpc(1 + \bar{c}^2)\}}{|c|^2\,\text{Re}\,(\lambda Xp)} + z^2\right)}{[1 + A_1 z + A_2 z^2 + A_1 z^3 + z^4]}$$

$$= \frac{(1 - 2tz + z^2)(1 - z^2)}{(1 - z^2)^2} \qquad \ldots(52)$$

where we have made use of the fact that the zeros of $B(z)$ on $|z| = 1$ are of even order. Comparing the coefficients of different powers of z in the numerator and the denominator on both sides of (52), we obtain

$$\begin{cases} \dfrac{\operatorname{Re}\{\lambda X p c(1+\bar{c}^2)\}}{|c|^2 \operatorname{Re}(\lambda X p)} = 2t \\ A_1 = -2\operatorname{Re}\left(\dfrac{1+c^2}{c}\right) + \dfrac{\operatorname{Re}\{\lambda X(1-c^2)\bar{c}\}}{|c|^2 \operatorname{Re}(\lambda X p)} = 0 \\ A_2 = 2 + \left|\dfrac{1+c^2}{c}\right|^2 - \dfrac{\operatorname{Re}\{\lambda X(1+\bar{c}^2)(1-c^2)\}}{|c|^2 \operatorname{Re}(\lambda X p)} = 2. \end{cases} \qquad \ldots(53)$$

Equations (53) lead to the single equation

$$\operatorname{Re}\left(\lambda X \frac{1-\bar{c}^2}{\bar{c}}\right) = 0. \qquad \ldots(54)$$

On replacing X by (38) and $p(c) = \dfrac{1-2tc+c^2}{1-c^2}$, we have from (54)

$$(1+a_1)^2 \operatorname{Re}\left(\lambda \frac{1+c}{1-c}\right) t^2 - 2(1+a_1)\operatorname{Re}\left\{\lambda \frac{1+c}{1-c}\left(\frac{1+\bar{c}^2}{\bar{c}} + 1 - a_1\right)\right\} t$$
$$+ \operatorname{Re}\left\{\lambda \frac{1+c}{1-c}\left(\frac{1+\bar{c}^2}{\bar{c}} + 1 - a_1\right)^2\right\} = 0.$$

If $\operatorname{Re}\left(\lambda \dfrac{1+c}{1-c}\right) \neq 0$, t is either t_1 or t_2, and if $\operatorname{Re}\left(\lambda \dfrac{1+c}{1-c}\right) = 0$, we have $t = t_3$.

For $\operatorname{Im} c \geqslant 0$ the values t_1 and t_2 when substituted in (41), yield respectively $B(z) \geqslant 0$ and $B(z) \leqslant 0$ on $|z| = 1$. Hence from Lemma 1 the value t_1 corresponds to minimum and t_2 corresponds to maximum value of $\operatorname{Re} F(p_1(c))$.

For $\operatorname{Re}\left(\lambda \dfrac{1+c}{1-c}\right) = 0$, by using Lemma 1 the value t_3 gives the minimum or maximum according as $\operatorname{Re}\lambda \leqslant 0$ or $\operatorname{Re}\lambda \geqslant 0$. Further from Lemma 3, extremal function $p(z) = \dfrac{1-2tz+z^2}{1-z^2}$, $-1 \leqslant t \leqslant 1$ will give minimum $\operatorname{Re} F(p_1(c))$ if $\operatorname{Re}(\lambda X p) \geqslant 0$, where X is given by (38). Now for $t = -1$ and $t = 1$ we have from $\operatorname{Re}(\lambda X p) \geqslant 0$, $\operatorname{Re}\left(\lambda \dfrac{c}{1-c^2}\right) \geqslant 0$ and $\operatorname{Re}\left\{\lambda \dfrac{c(1-c^2)}{(1-2a_1 c+c^2)^2}\right\} \geqslant 0$ respectively. It is easy to see that if $\operatorname{Im} c \geqslant 0$ and $\operatorname{Re}\left(\lambda \dfrac{c}{1-c^2}\right) \geqslant 0$, $0 \leqslant \arg\lambda \leqslant \pi/2$, we have $\operatorname{Re}\left(\lambda \dfrac{1+c}{1-c}\right) > 0$. Hence from $-1 \leqslant t_1 \leqslant 1$ we obtain (24).

Now $t_3 = \lim_{\mathrm{Re}\left(\lambda \frac{1+c}{1-c}\right) \to 0} t_2$ and from $-1 \leqslant t_2 \leqslant 1$ we obtain (25).

Proof of case II is similar to that of case I, but in this case we always have $\mathrm{Re}\left(\lambda \dfrac{1-c}{1+c}\right) > 0$.

Proof of case III — The extremal function $p_1(z) = \dfrac{1 + 2a_1 z + z^2}{1 - z^2}$ corresponds to function $p(z) = \dfrac{1+z}{1-z}$. We shall prove the case of minimum only. From Lemma 1, $p(z) = \dfrac{1+z}{1-z}$ will give minimum $\mathrm{Re}\, F(p_1(c))$ if $B(z) \geqslant 0$ on $|z| = 1$.

or

$$\mathrm{Re}\,(\lambda X p)\left[\frac{1}{z^2} + \frac{A_1}{z} + A_2 + A_1 z + z^2\right] \leqslant 0 \text{ on } |z| = 1. \qquad \ldots(55)$$

Substituting the values of A_1, A_2 and X from (42) and (38) in (55), we get

$$\left\{\left(z + \frac{1}{z}\right)\mathrm{Re}\left(\lambda \frac{c}{1-c^2}\right) - \mathrm{Re}\left(\lambda \frac{c}{1-c^2}\frac{1+\bar{c}^2}{\bar{c}}\right)\right\}\left(\frac{1}{z} - 2 + z\right)$$
$$\leqslant 0 \text{ on } |z| = 1$$

or equivalently

$$\left(z + \frac{1}{z}\right)\mathrm{Re}\left(\lambda \frac{c}{1-c^2}\right) - \mathrm{Re}\left(\lambda \frac{c}{1-c^2}\frac{1+\bar{c}^2}{\bar{c}}\right) \geqslant 0 \text{ on } |z| = 1. \qquad \ldots(56)$$

As $\min\left(z + \dfrac{1}{z}\right) = -2$ on $|z| = 1$.

(56) yields for $\mathrm{Re}\left(\lambda \dfrac{c}{1-c^2}\right) \geqslant 0$

$$\frac{1}{\mathrm{Re}\left(\lambda \frac{1+c}{1-c}\right)}\left[\mathrm{Re}\left(\lambda \frac{c}{1-c^2}\right) - \frac{|\lambda|(1-|c|^2)\,\mathrm{Im}\, c}{|1-c^2|\;|1+c|^2}\right]$$
$$\times\left[\mathrm{Re}\left(\lambda \frac{c}{1-c^2}\right) + \frac{|\lambda|(1-|c|^2)\,\mathrm{Im}\, c}{|1-c^2|\;|1+c|^2}\right] \leqslant 0.$$

Since $\mathrm{Re}\left(\lambda \dfrac{1+c}{1-c}\right) > 0$ for $\mathrm{Im}\, c \geqslant 0$ and $\mathrm{Re}\left(\lambda \dfrac{c}{1-c^2}\right) \geqslant 0$, we finally have

$$\mathrm{Re}\left(\lambda \frac{c}{1-c^2}\right) \leqslant \frac{|\lambda|(1-|c|^2)\,\mathrm{Im}\, c}{|1-c^2|\;|1+c|^2}. \qquad \ldots(57)$$

Similarly if $\text{Re}\left(\lambda \frac{c}{1-c^2}\right) \leqslant 0$, we have from (56)

$$-\frac{|\lambda|(1-|c|^2)\,\text{Im}\,c}{|1-c^2|\;|1-c|^2} \leqslant \text{Re}\left(\lambda \frac{c}{1-c^2}\right). \qquad ...(58)$$

(57) and (58) yield (28).

Proof of case IV is similar to that of case III.

This completes the proof of Theorem 3.

In Theorem 3, we have made use of the fact that $\arg \lambda \in [0, \pi/2]$. For other values of $\arg \lambda$, proof follows from Theorem 3.

Figures 1 and 2 show how the upper half semi-circle is divided into sub-regions corresponding to different extremal functions. The number in a region corresponds to the inequality defining that region in the statement of the theorem.

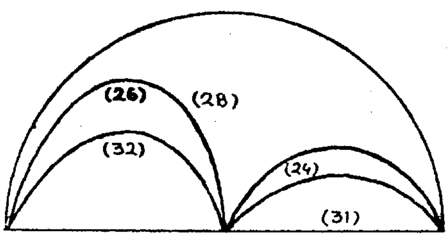

FIG. 1. [Minimum Re $F(p_1(c))$]

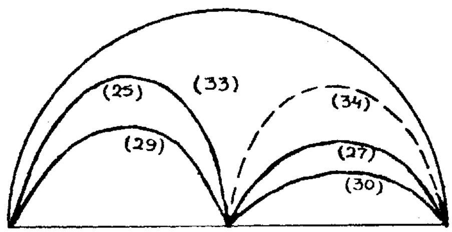

FIG. 2. [Maximum $F(p_1(c))$]

Corollary 2 — Let $z \in E$, $0 \leqslant \arg z \leqslant \pi/2$ and $p_1(z) \in P_1$, then

$$\text{Re}\, p_1(z) \geqslant \begin{cases} \dfrac{(1-|z|^2)(1-2a_1\,\text{Re}\,z+|z|^2)}{|1-2a_1 z+z^2|^2} \\ \qquad \text{if } \left|\dfrac{1+z}{1-z}\right| \text{Re}\, z - \dfrac{2|z|^2(1+a_1)}{|1-z^2|} \geqslant \text{Im}\, z \\ \phi_1(z, a_1) \text{ if } \left|\dfrac{1+z}{1-z}\right| \text{Re}\, z - \dfrac{2|z|^2(1+a_1)}{|1-z^2|} \\ \qquad \leqslant \text{Im}\, z \leqslant \left|\dfrac{1+z}{1-z}\right| \text{Re}\, z \\ \dfrac{(1-|z|^2)(1+2a_1\,\text{Re}\,z+|z|^2)}{|1-z^2|^2} \text{ if } \text{Im}\, z \geqslant \left|\dfrac{1+z}{1-z}\right| \text{Re}\, z \end{cases} \qquad ...(59)$$

and

$$\operatorname{Re} p_1(z) \leqslant \begin{cases} \dfrac{(1-|z|^2)(1+2a_1 \operatorname{Re} z+|z|^2)}{|1-z^2|^2} & \text{if } \operatorname{Im} z \leqslant \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z \\ \phi_1(-z,-a_1) & \text{if } \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z + \dfrac{2|z|^2(1-a_1)}{|1-z^2|} \\ & \quad \geqslant \operatorname{Im} z \geqslant \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z \\ \dfrac{(1-|z|^2)(1-2a_1 \operatorname{Re} z+|z|^2)}{|1-2a_1 z+z^2|^2} & \text{if } \dfrac{1-z}{1+z} \operatorname{Re} z \\ & \quad + \dfrac{2|z|^2(1-a_1)}{|1-z^2|} \leqslant \operatorname{Im} z \end{cases} \quad \ldots(60)$$

where

$$\phi_1(z, a_1) = \frac{1-|z|^2}{|1-z|^2}\left[1 - \frac{(1-a_1)|z|^2}{\{|1-z^2|+2 \operatorname{Im} z\} \operatorname{Im} z}\right].$$

These inequalities are sharp.

Corollary 3 — Let $z \in E$, $0 \leqslant \arg z \leqslant \pi/2$ and $f(z) \in S_1^*$, then

$$|f(z)| \geqslant \begin{cases} \dfrac{|z|}{|1-2a_1 z+z^2|} & \text{if } \left|\dfrac{1+z}{1-z}\right| \operatorname{Re} z - \dfrac{2|z|^2(1+a_1)}{|1-z^2|} \geqslant \operatorname{Im} z \\ \phi_2(z,a_1) & \text{if } \left|\dfrac{1+z}{1-z}\right| \operatorname{Re} z - \dfrac{2|z|^2(1+a_1)}{|1-z^2|} \\ & \quad \leqslant \operatorname{Im} z \leqslant \left|\dfrac{1+z}{1-z}\right| \operatorname{Re} z \\ \dfrac{|z|}{|1-z|^{(1+a_1)}\,|1+z|^{(1-a_1)}} & \text{if } \operatorname{Im} z \geqslant \left|\dfrac{1+z}{1-z}\right| \operatorname{Re} z \end{cases} \quad \ldots(61)$$

and

$$|f(z)| \leqslant \begin{cases} \dfrac{|z|}{|1-z|^{(1+a_1)}\,|1+z|^{(1-a_1)}} & \text{if } \operatorname{Im} z \leqslant \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z \\ \phi_2(-z,-a_1) & \text{if } \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z + \dfrac{2|z|^2(1-a_1)}{|1-z^2|} \\ & \quad \geqslant \operatorname{Im} z \geqslant \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z \\ \dfrac{|z|}{|1-2a_1 z+z^2|} & \text{if } \left|\dfrac{1-z}{1+z}\right| \operatorname{Re} z + \dfrac{2|z|^2(1-a_1)}{|1-z^2|} \leqslant \operatorname{Im} z \end{cases} \quad \ldots(62)$$

where

$$\phi_2(z, a_1) = \frac{|z|}{|1-z|^2}\left[\frac{|1-z|^2}{(1-|z|)^{(|z|+\operatorname{Re} z)/|z|}(1+|z|)^{(|z|-\operatorname{Re} z)/|z|}}\right.$$

$$\times \left\{\frac{|1-z^2|-2\operatorname{Im} z}{|1+z^2|+2\operatorname{Im} z}\right\}^{\frac{1}{2}} \left\{\frac{|1-z^2|\,|z|-(1+|z|^2)\operatorname{Im} z}{|1-z^2|\,|z|+(1+|z|^2)\operatorname{Im} z}\right.$$

$$\left.\times \frac{|z|+\operatorname{Im} z}{|z|-\operatorname{Im} z}\right\}^{\operatorname{Re} z/2|z|}\left.\right]^{(1-a_1)|z|^2/2(\operatorname{Im} z)^2}.$$

First and third inequalities of (61) and (62) are sharp.

Proof of corollary 3 follows from Corollary 2 by using the relation (1).

Theorem 4 — Let $c \in E$, $c \neq 0$, $\operatorname{Im} c \geqslant 0$ be fixed. If $f(z) \in T_1$ gives the extreme value of $\operatorname{Re}\{e^{i\alpha} f(c)\}$, $0 \leqslant \alpha \leqslant \pi/2$, then $f(z)$ has either the form

$$f(z) = \frac{z}{(1-z)^2}\,\frac{1+z^2-(1-a_1)z-(1+a_1)zt}{1+z^2+(1-a_1)z-(1+a_1)zt},\ -1 \leqslant t \leqslant 1 \qquad \ldots(63)$$

or

$$f(z) = \frac{z}{(1+z)^2}\,\frac{1+z^2+(1+a_1)z-(1-a_1)zt}{1+z^2-(1+a_1)z-(1-a_1)zt},\ -1 \leqslant t \leqslant 1. \qquad \ldots(64)$$

The following cases arise :

Case I — The function defined by (63) gives the minimum $\operatorname{Re}[e^{i\alpha} f(c)]$ where

$$t = t_1 = \frac{\operatorname{Re}\left\{e^{i\alpha}\dfrac{c}{(1-c)^2}\left(\dfrac{1+\bar{c}^2}{\bar{c}}+1-a_1\right)\right\} - \dfrac{1-|c|^2}{|1-c|^2\,|c|}\operatorname{Im} c}{(1+a_1)\operatorname{Re}\left(e^{i\alpha}\dfrac{c}{(1-c)^2}\right)}$$

if

$$\frac{(1-|c|^2)\,|c|\operatorname{Im} c}{|1-c^2|^2\,|1+c|^2} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}\operatorname{Re}\left(e^{i\alpha}\frac{c}{(1-c)^2}\right)$$

$$\geqslant \operatorname{Re}\left(e^{i\alpha}\frac{c^2}{(1-c^2)^2}\right) \geqslant \frac{(1-|c|^2)\,|c|\operatorname{Im} c}{|1-c^2|^2\,|1+c|^2} \qquad \ldots(65)$$

and it gives the maximum where

$$t = t_2 = \frac{\operatorname{Re}\left\{e^{i\alpha}\dfrac{c}{(1-c)^2}\left(\dfrac{1+\bar{c}^2}{\bar{c}}+1-a_1\right)\right\} + \dfrac{1-|c|^2}{|1-c|^2\,|c|}\operatorname{Im} c}{(1+a_1)\operatorname{Re}\left(e^{i\alpha}\dfrac{c}{(1-c)^2}\right)}$$

if

$$-\frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1+c|^2\operatorname{Re}\left(e^{i\alpha}\dfrac{c}{(1-c)^2}\right)}+2(1+a_1)\frac{|c|^2}{|1+c|^4}$$

$$\geqslant \frac{\operatorname{Re}\{e^{i\alpha}c^2/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}c/(1-c)^2\}} \geqslant -\frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1+c|^2\operatorname{Re}\{e^{i\alpha}c/(1-c)^2\}}. \quad ...(66)$$

Case II — The function defined by (64) gives the minimum Re $\{e^{i\alpha} f(c)\}$

where

$$t = t_4 = \frac{\operatorname{Re}\left\{e^{i\alpha}\dfrac{c}{(1+c)^2}\left(\dfrac{1+\bar{c}^2}{\bar{c}}-1-a_1\right)\right\}+\dfrac{1-|c|^2}{|1+c|^2|c|}\operatorname{Im} c}{(1-a_1)\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}},$$

if

$$-\frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1-c|^2\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}}-2(1-a_1)\frac{|c|^2}{|1-c|^4}$$

$$\leqslant \frac{\operatorname{Re}\{e^{i\alpha}c^2/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}}$$

$$\leqslant -\frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1-c|^2\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}} \quad ...(67)$$

and it gives the maximum where

$$t = t_5 = \frac{\operatorname{Re}\left\{e^{i\alpha}\dfrac{c}{(1+c)^2}\left(\dfrac{1+\bar{c}^2}{\bar{c}}-1-a_1\right)\right\}-\dfrac{1-|c|^2}{|1+c|^2|c|}\operatorname{Im} c}{(1-a_1)\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}}$$

if

$$\frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1-c|^2}-2(1-a_1)\frac{|c|^2}{|1-c|^4}\operatorname{Re}\left(e^{i\alpha}\frac{c}{(1+c)^2}\right)$$

$$\leqslant \operatorname{Re}\left(e^{i\alpha}\frac{c^2}{(1-c^2)^2}\right) \leqslant \frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1-c|^2}. \quad ...(68)$$

Case III — $f(z) = \dfrac{z(1+2a_1z+z^2)}{(1-z^2)^2}$ gives the minimum Re $\{e^{i\alpha} f(c)\}$ either if

$$\operatorname{Re}\left(e^{i\alpha}\frac{c^2}{(1-c^2)^2}\right) \leqslant \frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1+c|^2} \quad ...(69)$$

or if

$$\frac{\operatorname{Re}\{e^{i\alpha}c^2/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}} \geqslant -\frac{(1-|c|^2)|c|\operatorname{Im} c}{|1-c^2|^2|1-c|^2\operatorname{Re}\{e^{i\alpha}c/(1+c)^2\}} \quad ...(70)$$

and it gives the maximum either if

$$\frac{\operatorname{Re}\{e^{i\alpha} c^2/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha} c/(1-c)^2\}} \leqslant -\frac{(1-|c|^2)|c| \operatorname{Im} c}{|1-c^2|^2 |1+c|^2 \operatorname{Re}\{e^{i\alpha} c/(1-c)^2\}} \quad \ldots(71)$$

or if

$$\operatorname{Re}\{e^{i\alpha} c^2/(1-c^2)^2\} \geqslant \frac{(1-|c|^2)|c| \operatorname{Im} c}{|1-c^2|^2 |1-c|^2}. \quad \ldots(72)$$

Case IV — $f(z) = \dfrac{z}{1-2a_1 z + z^2}$ gives the minimum either if

$$\frac{(1-|c|^2)|c| \operatorname{Im} c}{|1-c^2|^2 |1+c|^2} + 2(1+a_1)\frac{|c|^2}{|1+c|^4} \operatorname{Re}\left(e^{i\alpha}\frac{c}{(1-c)^2}\right) \leqslant \operatorname{Re}\left(e^{i\alpha}\frac{c^2}{(1-c^2)^2}\right) \quad \ldots(73)$$

or if

$$-\frac{(1-|c|^2)|c| \operatorname{Im} c}{|1-c^2|^2 |1-c|^2 \operatorname{Re}\{e^{i\alpha} c/(1+c)^2\}} - 2(1-a_1)\frac{|c|^2}{|1-c|^4} \geqslant \frac{\operatorname{Re}\{e^{i\alpha} c^2/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha} c/(1-c)^2\}} \quad \ldots(74)$$

and it gives the maximum either if

$$-\frac{(1-|c|^2)|c| \operatorname{Im} c}{|1-c^2|^2 |1+c|^2 \operatorname{Re}\{e^{i\alpha} c/(1-c)^2\}} + 2(1+a_1)\frac{|c|^2}{|1+c|^4} \leqslant \frac{\operatorname{Re}\{e^{i\alpha} c^2/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha} c/(1-c)^2\}} \quad \ldots(75)$$

or if

$$\frac{(1-|c|^2)|c| \operatorname{Im} c}{|1-c^2|^2 |1-c|^2} - 2(1-a_1)\frac{|c|^2}{|1-c|^4} \operatorname{Re}\left[e^{i\alpha}\frac{c}{(1+c)^2}\right] \geqslant \operatorname{Re}\left(e^{i\alpha}\frac{c^2}{(1-c^2)^2}\right). \quad \ldots(76)$$

Theorem 5 — Let $c \in E$, $c \neq 0$, $\operatorname{Im} c \geqslant 0$ be fixed. If $g(z) \in I_1$ gives the extreme value of $\operatorname{Re}\{e^{i\alpha} g'(c)\}$, $0 \leqslant \alpha \leqslant \pi/2$, then $g(z)$ has either the form

$$g(z) = \frac{1}{(1-z)^2}\,\frac{1+z^2-(1-a_1)z-(1+a_1)zt}{1+z^2+(1-a_1)z-(1+a_1)zt}, \quad -1 \leqslant t \leqslant 1 \quad \ldots(77)$$

or

$$g(z) = \frac{1}{(1+z)^2} \frac{1 + z^2 + (1+a_1) z - (1-a_1) zt}{1 + z^2 - (1+a_1) z - (1-a_1) zt}, \; -1 \leqslant t \leqslant 1. \qquad \ldots(78)$$

The following cases arise :

Case I — The function defined by (77) gives the minimum Re $\{e^{i\alpha} g'(c)\}$

where

$$t = t_1 = \frac{\text{Re}\,[e^{i\alpha}\, c\{1 + \bar{c}^2 + \bar{c}(1-a_1)\}/(1-c)^2] - (1 - |c|^2)\,\text{Im}\, c/|1-c|^2}{|c|^2(1+a_1)\,\text{Re}\,\{e^{i\alpha}/(1-c)^2\}}$$

if

$$\frac{(1-|c|^2)\,\text{Im}\, c}{|1-c^2|^2\,|1+c|^2} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}\,\text{Re}\,\{e^{i\alpha}/(1-c)^2\}$$
$$\geqslant \text{Re}\,\{e^{i\alpha}\, c/(1-c^2)^2\} \geqslant \frac{(1-|c|^2)\,\text{Im}\, c}{|1-c^2|^2\,|1+c|^2} \qquad \ldots(79)$$

and it gives the maximum where

$$t = t_2 = \frac{\text{Re}\,[e^{i\alpha}\, c\{1 + \bar{c}^2 + \bar{c}(1-a_1)\}/(1-c)^2] + (1 - |c|^2)\,\text{Im}\, c/|1-c|^2}{|c|^2(1+a_1)\,\text{Re}\,\{e^{i\alpha}/(1-c)^2\}}$$

if

$$-\frac{(1-|c|^2)\,\text{Im}\, c}{|1-c^2|^2\,|1+c|^2\,\text{Re}\,\{e^{i\alpha}/(1-c)^2\}} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}$$
$$\geqslant \frac{\text{Re}\,\{e^{i\alpha}\, c/(1-c^2)^2\}}{\text{Re}\,\{e^{i\alpha}/(1-c)^2\}} \geqslant -\frac{(1-|c|^2)\,\text{Im}\, c}{|1-c^2|^2\,|1+c|^2\,\text{Re}\,\{e^{i\alpha}/(1-c)^2\}}. \qquad \ldots(80)$$

Case II — The function defined by (78) gives the minimum Re $\{e^{i\alpha} g'(c)\}$

where

$$t = t_4 = \frac{\text{Re}\,[e^{i\alpha}\, c\{1 + \bar{c}^2 - \bar{c}(1+a_1)\}/(1+c)^2] + (1 - |c|^2)\,\text{Im}\, c/|1+c|^2}{|c|^2(1-a_1)\,\text{Re}\,\{e^{i\alpha}/(1+c)^2\}}$$

if

$$-\frac{(1-|c|^2)\,\text{Im}\, c}{|1-c^2|^2\,|1-c|^2} - 2(1-a_1)\frac{|c|^2}{|1-c|^4}\,\text{Re}\,\{e^{i\alpha}/(1+c)^2\}$$
$$\leqslant \text{Re}\,\{e^{i\alpha}\, c/(1-c^2)^2\} \leqslant -\frac{(1-|c|^2)\,\text{Im}\, c}{|1-c^2|^2\,|1-c|^2} \qquad \ldots(81)$$

and it gives the maximum where

$$t = t_5 = \frac{\text{Re}\,[e^{i\alpha}\, c\{1 + \bar{c}^2 - \bar{c}(1+a_1)\}/(1+c)^2] - (1 - |c|^2)\,\text{Im}\, c/|1+c|^2}{|c|^2(1-a_1)\,\text{Re}\,\{e^{i\alpha}/(1+c)^2\}}$$

if

$$\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1-c|^2 \operatorname{Re}\{e^{i\alpha}/(1+c)^2\}} - 2(1-a_1)\frac{|c|^2}{|1-c|^4}$$
$$\leqslant \frac{\operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}/(1+c)^2\}} \leqslant \frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1-c|^2 \operatorname{Re}\{e^{i\alpha}/(1+c)^2\}} \quad \ldots(82)$$

Case III — $g(z) = \dfrac{1+2a_1z+z^2}{(1-z^2)^2}$ gives the minimum if

$$\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1+c|^2} \geqslant \operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\} \geqslant -\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1-c|^2} \quad \ldots(83)$$

and it gives the maximum either if

$$\frac{\operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}/(1-c)^2\}} \leqslant -\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1+c|^2 \operatorname{Re}\{e^{i\alpha}/(1-c)^2\}} \quad \ldots(84)$$

or if

$$\frac{\operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}/(1+c)^2\}} \geqslant \frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1-c|^2 \operatorname{Re}\{e^{i\alpha}/(1+c)^2\}}. \quad \ldots(85)$$

Case IV — $g(z) = \dfrac{1}{(1-2a_1z+z^2)}$ gives the minimum if

$$\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1+c|^2} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}\operatorname{Re}\{e^{i\alpha}/(1-c)^2\}$$
$$\leqslant \operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\} \leqslant -\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1-c|^2}$$
$$-2(1-a_1)\frac{|c|^2}{|1-c|^4}\operatorname{Re}\{e^{i\alpha}/(1+c)^2\} \quad \ldots(86)$$

and it gives the maximum either if

$$-\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1+c|^2 \operatorname{Re}\{e^{i\alpha}/(1-c)^2\}} + 2(1+a_1)\frac{|c|^2}{|1+c|^4}$$
$$\leqslant \frac{\operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}/(1-c)^2\}} \quad \ldots(87)$$

or if

$$\frac{(1-|c|^2)\operatorname{Im} c}{|1-c^2|^2|1-c|^2 \operatorname{Re}\{e^{i\alpha}/(1+c)^2\}} - 2(1-a_1)\frac{|c|^2}{|1-c|^4}$$
$$\geqslant \frac{\operatorname{Re}\{e^{i\alpha}c/(1-c^2)^2\}}{\operatorname{Re}\{e^{i\alpha}/(1+c)^2\}}. \quad \ldots(88)$$

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