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ON SOME UNIVALENT FUNCTIONS IN THE UNIT DISC

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Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an analytic function over $D(\,|z| < 1)$. For such a function the concept of starlikeness with respect to N-symmetric points or simply N-starlikeness is introduced. Sakaguchi (1959) considered the case $N = 2$. We extend Sakaguchi's result (1959, Th. 1, p. 72) for arbitrary N and show that his result can be obtained under weaker conditions. Moreover, an investigation of zeros of the function $\sum_{p=0}^{N-1} \varepsilon^{-p} f(\varepsilon^p z)$, where $\varepsilon = \exp(2\pi i/N)$ has also been undertaken.

An analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \ldots(1)$$

over $D(\,|z| < 1)$ is called (Sakaguchi 1959) starlike with respect to symmetric points if for every r close to $1, r < 1$ and every z_0 on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at $z = z_0$ as z transverses the circle $|z| = r$ in the positive direction, i.e.,

$$\text{Re}\,\frac{zf'(z)}{f(z) - f(-z_0)} > 0 \qquad \ldots(2)$$

for $z = z_0$, $|z_0| = r$.

One of the interesting results proved by Sakaguchi (1959) is as follows :

Theorem A — A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent and starlike with respect to symmetric points if, and only if,

$$\text{Re}\,\frac{zf'(z)}{f(z) - f(-z)} > 0 \text{ for } |z| < 1. \qquad \ldots(3)$$

Seeing (2), one is tempted to extend the definition of starlikeness with respect to symmetric points to starlikeness with respect to N-symmetric points in the following way :

Let

$$f_1(z) \equiv 0 \qquad \ldots(4)$$

$$f_N(z) = \frac{\sum\limits_{p=0}^{N-1} \epsilon^{-p} f(\epsilon^p z)}{\sum\limits_{p=1}^{N-1} \epsilon^{-p}} = - \sum_{p-1}^{N-1} \epsilon^{-p} f(\epsilon^p z) \quad (N \geqslant 2) \qquad \ldots(5)$$

where $\epsilon = \exp(2\pi i/N)$.

The functions $f_N(z)$ may be called the N-weighted mean functions of $f(z)$.

Definition 1 — A function $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ analytic in $|z| < 1$ is said to be N-starlike if for every r close to 1, $r < 1$, the angular velocity of $f(z)$ about the point $f_N(z_0)$ is positive at $z = z_0$ as z transverses the circle $|z| = r$ in the positive direction i.e.

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f_N(z_0)} > 0 \text{ for } z = z_0, \ |z_0| = r. \qquad \ldots(6)$$

For $N = 2$, (6) is nothing else but (2). In the present paper we prove the following :

Theorem 1 — $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, analytic over D, is conformal and N-starlike if and only if,

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f_N(z)} > 0 \text{ over } D. \qquad \ldots(7)$$

PROOF : Let $f(z)$ be conformal and N-starlike over D. Then, by (5), for every z sufficiently close to the boundary of D,

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f_N(z)} > 0$$

and, therefore,

$$\operatorname{Re} \frac{f(z) - f_N(z)}{zf'(z)} > 0 \qquad \ldots(8)$$

for z sufficiently close to the boundary of D. Since $f(z) - f_N(z) = 0$ at $z = 0$ and $f'(z) \neq 0$ over D, one can, at once, say that $\dfrac{f(z) - f_N(z)}{zf'(z)}$ is analytic over D. In view of the fact that a non-constant harmonic function over a given domain never assumes minimum value at an interior point, (8) implies that

$$\operatorname{Re} \frac{f(z) - f_N(z)}{zf'(z)} > 0 \quad \text{over } D \qquad \ldots(9)$$

and, therefore,

$$\mathrm{Re}\,\frac{zf'(z)}{f(z) - f_N(z)} > 0 \quad \text{over } D. \qquad ...(10)$$

This establishes that condition (7) is necessary.

Sufficiency of condition (7) follows from the following theorem due to Sakaguchi (1959, p. 73).

Theorem B — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic over D. If

$$\mathrm{Re}\,\frac{zf'(z)}{\sum_{p=0}^{N-1} \epsilon^{-p} f(\epsilon^p z)} > 0 \quad \text{over } D \qquad ...(11)$$

then $f(z)$ is univalent over D.

(11) is nothing else but (7) as $f(z) - f_N(z) = \sum_{p=0}^{N-1} \epsilon^{-p} f(\epsilon^p z)$. Moreover, univalent functions are primarily conformal. Hence the proof of the theorem is complete.

Note : From Theorem 1, it follows that $f(z) - f_N(z) \neq 0$ for $0 < |z| < 1$ is a necessary condition for $f(z)$ to be N-starlike over D. Hence, it becomes necessary to study the zeros of $f(z) - f_N(z)$ in $0 < |z| < 1$, assuming $f(z)$ to be univalent over D. For $N = 1, 2$, univalency is sufficient to declare $f(z) - f_N(z) \neq 0$ for $0 < |z| < 1$.

For general N, we have the following :

Theorem 2 — Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $(a_1 \neq 0)$ be a univalent starlike function of order $\beta \geqslant \frac{1}{2}$ i.e.

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \beta \text{ for } |z| < 1. \qquad ...(12)$$

Let $\epsilon = \exp(2\pi i/N)$. Then

$$\sum_{k=0}^{N-1} \epsilon^{-k} f(\epsilon^k z) \neq 0 \qquad ...(13)$$

for any positive integer N and $0 < |z| < 1$.

PROOF : To prove (13), it will be sufficient to show that for any $z \neq 0$ in D, the points $f(z)$, $\epsilon^{-1} f(\epsilon z)$, ..., $\epsilon^{-N+1} f(\epsilon^{N-1} z)$ belong to the same half plane bounded by a straight line passing through origin and at least one point is not on this line.

Let $z \neq 0$ be an arbitrary point in $|z| < 1$. Without loss of generality we can assume $\arg f(z) = 0$ for, otherwise, we may consider the function

$$g(t) = \exp(-i \arg f(z)) f(t) \qquad \ldots(14)$$

which is again a starlike function of order β and satisfies the condition $\arg g(z) = 0$.

Let us define

$$A = \{A_k : A_k = \epsilon^{-k} f(\epsilon^k z)\ k = 0, 1, \ldots, N-1\} \qquad \ldots(15)$$

$$\alpha_k = \arg(\epsilon^{-k} f(\epsilon^k z)),\ \ k = 0, 1, \ldots, N-1 \qquad \ldots(16)$$

One can easily claim that

$$\arg f(\epsilon^{k+1} z) - \arg f(\epsilon^k z) > \pi/N \ \ (k = 0, 1, \ldots, N-1) \qquad \ldots(17)$$

because, from (12),

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) = \text{Re}\ re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} > \beta \geqslant \tfrac{1}{2}. \qquad \ldots(18)$$

Thus, one has

$$\alpha_{k+1} - \alpha_k > -\pi/N \ \text{ for } \ k = 0, 1, \ldots, N-1. \qquad \ldots(19)$$

On the other hand, $f(z)$ univalent implies that for every $k = 0, \ldots N-1$,

$$\arg f(\epsilon^k z) + \sum_{p=k}^{N-1} \{\arg f(\epsilon^{p+1} z) - \arg f(\epsilon^p z)\} = 2\pi,$$

and, hence, using (16) and (17), we have

$$\alpha_k = 2\pi - \sum_{p=k}^{N-1} \{\arg f(\epsilon^{p+1} z) - \arg f(\epsilon^p z)\} - \frac{2k\pi}{N}$$

$$< 2\pi - \frac{\pi}{N}(N-k) - \frac{2k\pi}{N} = \frac{N-k}{N}\pi. \qquad \ldots(20)$$

Combining (20) with (19), we have

$$-\frac{\pi}{N} + \alpha_{k-1} < \alpha_k < \frac{N-k}{N}\pi \ \text{ for } \ k = 1, 2, \ldots N-1. \qquad \ldots(21)$$

Let us denote

$$I_p = \left[\frac{N-p-1}{N}\pi, \frac{N-p}{N}\pi\right) \text{ for } p = 1, 2, \ldots N-1$$

and

$$I_N = \left(-\frac{\pi}{N}, 0\right\}.$$

Then

$$\alpha_1 \in \bigcup_{p=1}^{N} I_p = \left(-\frac{\pi}{N}, \frac{N-1}{N}\pi\right)$$

follows in view of (21). Suppose that

$$\alpha_1 \in I_p,\ p \leqslant N-1. \quad \ldots(22)$$

Denote

$$\beta_p \overset{def}{=} \max_{1 \leqslant k \leqslant p} \alpha_k = \alpha_j, \text{ say}. \quad \ldots(23)$$

Using (21) and (22) it easily follows that

$$\alpha_k \leqslant \alpha_j \text{ for } 0 \leqslant k \leqslant N-1. \quad \ldots(24)$$

Also, for $k < j$,

$$\alpha_k > \alpha_j - \frac{k-j}{N} > \alpha_j - \pi \quad \ldots(25)$$

and for $k < j$

$$\alpha_k > -\frac{k}{N}\pi > \frac{N-j}{N}\pi - \pi > \alpha_j - \pi \quad \ldots(26)$$

follows from (21). Hence, in view of (23), (24), (25) and (26), it readily follows that

$$A \subset H_{\beta_p} = \{w : \beta_p - \pi \leqslant \arg w \leqslant \beta_p\}$$

Furthermore, since in view of (22), $\beta_p > 0$, A_0 is an interior point to this half plane.

If $\alpha_1 \in I_N$ and $\beta_N = \max\{0, \beta_{N-1}\}$, then, on similar lines, one can show that

$$A \subset H_\beta = \{w : \beta_N - \pi \leqslant \arg w \leqslant \beta_N\}$$

and at least one of the points A_0 or A_1 is an interior point to the half plane. This establishes the theorem.

Note : We actually showed that exactly one point, namely $\alpha_j = \beta_p$ lies on the boundary of H_{β_p}.

Hence, we can extend our result by writing $\sum_{k=0}^{p} \epsilon^{-k} f(\epsilon^k z) \neq 0$ for every positive integer p.

Corollary 1 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a univalent and convex function in $|z| < 1$. Then, for $0 < |z| < 1$,

$$\sum_{k=0}^{p} \epsilon^{-k} f(\epsilon^k z) \neq 0$$

for every positive p; $\epsilon = \exp(2\pi i/N)$, $N > 1$.

PROOF : Since every univalent convex function is starlike of order $\frac{1}{2}$ (Marx 1932), the result immediately follows in view of Theorem 2.

Theorem 3 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a univalent function with real coefficients satisfying

(i) $a_2 \geqslant 0$

(ii) $a_{n+1} - a_n > A \geqslant \sqrt{\frac{1}{3}}$ $(n = 2, 3, \ldots)$.

Then there exists a positive integer $N_0 = N_0(A)$ such that when $N > N_0$, we have

$$\sum_{p=0}^{N-1} \epsilon^{-p} f(\epsilon^p z) = 0 \qquad \ldots(27)$$

at least at $N + 1$ distinct points in $|z| < 1$. $\epsilon = \exp(2\pi i/N)$.

PROOF : One can easily see that

$$g_N(z) \stackrel{def}{=} \frac{1}{N} \sum_{p=0}^{N-1} \epsilon^{-p} f(\epsilon^p z)$$

$$= z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \ldots \qquad \ldots(28)$$

and

$$g_N(\epsilon z) = \epsilon\, g_N(z) \text{ for every } z \ (|z| < 1). \qquad \ldots(29)$$

Hence, in view of (29) the result will stand true if we show the existence of one point z^*, $(0 < |z^*| < 1)$ such that

$$g_N(z^*) = 0.$$

Let a positive integer N be fixed. It can be written as

$$N = 2^p M. \qquad \ldots(30)$$

Where p is integer $\geqslant 0$ and $M > 0$ is an odd integer. With the above representation of N, we consider the function

$$\tilde{g}_N(z) = \exp(-i\pi/2^p) \,.\, g_N(z e^{i\pi/2^p})$$

$$= z + \sum_{j=1}^{\infty} (-1)^j a_{Nj+1} z^{Nj+1}. \qquad \ldots(31)$$

The proof of the theorem will be complete if we show that $\tilde{g}_N(z) = 0$ at some point in the interval (0, 1).

One can easily check that for $z > 0$, and sufficiently small,

$$\tilde{g}_N(z) > 0. \qquad \ldots(32)$$

Hence, we are required to show the existence of a point z_0 $(0 < z_0 < 1)$ such that

$$\tilde{g}_N(z_0) \leqslant 0. \qquad \ldots(33)$$

Let

$$h_N(z) = a_{N+1}z^N - a_{2N+1}z^{2N} \qquad (0 \leqslant z \leqslant 1).$$

One can easily check that $h_N(z)$ is maximum at

$$z_1 = \frac{a_{N+1}}{2a_{2N+1}}. \qquad \ldots(34)$$

Since $f(z)$ is univalent with real coefficients, we have [Nehari 1952, p. 219]

$$|\, a_n \,| \leqslant n \ \text{ for } \ n = 2, 3, 4, \ldots \qquad \ldots(35)$$

In view of (34), (35) and conditions (i) and (ii) one can readily show the existence of a positive integer $N_1 = N_1(A)$ such that

$$\max_{0<z<1} h_N(z) = h_N(z_1) = \frac{a_{N+1}^2}{4a_{2N+1}} > \frac{(N-1)^2 A^2}{4(2N+1)} > 1 \qquad \ldots(36)$$

for all $N > N_1(A)$.

Let

$$S(N, p) = 2a_{Np+1} \cdot a_{2N+1} - a_{N(p+1)+1} \cdot a_{N+1} \quad (p \text{ integer} \geqslant 3) \qquad \ldots(37)$$

In view of (35) and the conditions (i) and (ii) we have $S(N, p) > 0$ if

$$2(Np - 1)(2N - 1)A^2 - (Np + N + 1)(N + 1) \geqslant 0 \qquad \ldots(38)$$

i.e., if

$$T(N, p, A) \overset{def}{=} (4pA^2 - p - 1)N^2 - (p + 2)(1 + 2A^2)N + 2A^2 - 1 > 0.$$

We observe that $T(N, p, A) > 0$ only if $A^2 > (p + 1)/4p$ which is automatically satisfied since, from condition (ii),

$$A^2 > \tfrac{1}{3} \geqslant (p + 1)/4p \qquad (p > 3).$$

We now observe that $T(N, p, A)$ is first order polynomial in p. Hence it will be an increasing function of p if the coefficient of p is positive and that is possible if

$$N > N_2(A) = \left[\frac{1 + 2A^2}{4A^2 - 1}\right] \qquad \ldots(39)$$

If we assume $N > N_2(A)$, then

$$\min_{p \geqslant 3} T(N, p, A) = T(N, 3, A) = 4(3A^2 - 1)\, N^2 - 5(1 + 2A^2)\, N + (2A^2 - 1) \geqslant 0$$

if

$$N > N_3(A) = \left[\frac{1}{8(3A^2 - 1)} \{5(1 + 2A^2) + (4A^4 + 20A^2 + 9)^{1/2}\}\right].$$

It then follows that $T(N, p, A) \geqslant 0$ and hence

$$S(N, p) > 0 \qquad \ldots(40)$$

for $p > 3$, $A^2 > \frac{1}{3}$ and $N > \max\{N_2(A), N_3(A)\}$.

From (36) and (40), we finally obtain

$$\tilde{g}_N(z_1) = z_1(1 - h_N(z_1)) - \tfrac{1}{2} \cdot \sum_{p=3,5,7,\ldots} \frac{S(N, p)}{a_{2N+1}}\, z_1^{Np+1} < 0$$

for all $N > N_0(A) = \max\{N_1(A), N_2(A), N_3(A)\}$ and this is what we wanted to show to declare the result true.

Corollary 2 — Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfy the hypothesis of the Theorem 3. Then $f(z)$ cannot be N-starlike for $N > N_0(A)$.

Note : From conditions (i), (ii) and (35), one can easily observe that $A \leqslant 1$. It follows that $N_1(A) > 10$ and $N_2(A) \leqslant 9$ and, therefore $N_0(A) = \max\{N_1(A), N_3(A)\}$.

We can also obtain a better estimate for $N_0(A)$ if we take the value of $a_2 > 0$ into account. In particular, we have the result that if $f(z)$ is Koebe function $f(z) = \dfrac{z}{(1 - z)^2}$, then $f_N(z)$ fails to be nonzero in $0 < |z| < 1$ and, therefore, $f(z)$ fails to be N-starlike for any $N > 7$.

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