Indian J. pure appl. Math., **12**(4) : 521–529, April 1981

ON A BANACH SPACE OF A CLASS OF DIRICHLET SERIES

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(*Received* 1 *April* 1980)

Few subclasses of functions represented by Dirichlet series whose coefficients satisfy certain simple conditions, with suitable definition of compositions and norm, seem to be well behaved as Banach spaces. Ω_0 is one of them. It has been shown in the following that it is a nonuniformly convex Banach space. A characterization of linear functionals has been obtained in a very simple way. A characterization of bounded linear transformation in terms of matrix has also been obtained.

1. INTRODUCTION

Consider the Dirichlet series $\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ where

$$0 \leqslant \lambda_1 < \lambda_2 < \lambda_3 \ldots \lambda_n \to \infty (n \to \infty),$$

$s = \sigma + it(\sigma, t$ are real variables) and $\{a_n\}$ is a sequence of complex numbers. It is well known that this series converges in a left half-plane and that the sum function $f(s)$ is holomorphic in its region of convergence (Mandelbrojt 1944).

Let $\sum_{n=1}^{\infty} \alpha_n e^{s\lambda_n} = u(s)$, be a fixed Dirichlet series having none of the α_n's equal to zero and the exponents λ_n satisfy the condition $\lim_{n\to\infty} \frac{\log n}{\lambda_n} = D = 0$, so that its abscissa of absolute convergence (σ_a^u) coincides with its abscissa of ordinary convergence (σ_c^u) (Mandelbrojt 1944). Assume σ_a^u to be greater than or equal to α, then the left half plane $\sigma < \alpha$ is the region of convergence R, for u.

Let Ω_0 be the class of all functions f, $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ is a Dirichlet series having the same sequence λ_n of exponents as that of $u(s)$ and $\left|\frac{a_n}{\alpha_n}\right| \to 0 (n \to \infty)$. It is easy to see that if σ_*^u be greater than or equal to α every element of Ω_0 has its abscissa of absolute convergence $\geqslant \alpha$, but it does not include all functions whose abscissa of absolute convergence is $\geqslant \alpha$. Also if u is an entire function, Ω_0 includes entire functions only. However, it does not include all entire functions.

Now we define compositions namely, 'addition (+)' 'scalar multiplication (.)' and $\| ; \|$ for the elements of Ω_0 as follows :

$$(f + g)(s) = \sum_{n=1}^{\infty} (a_n + b_n)\, e^{s\lambda_n} \qquad \text{...(1.1)}$$

$$(t \,.\, f)(s) = \sum_{n=1}^{\infty} (t a_n)\, e^{s\lambda_n} \qquad \text{...(1.2)}$$

$$\| f \| = \sup_n \left| \frac{a_n}{\alpha_n} \right| \qquad \text{...(1.3)}$$

where $f, g \in \Omega_0$ such that $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, $g(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$ and $t \in C$.

It is easy to verify that under these operations Ω_0 becomes a normed linear space. The object of the present paper is to study the properties of Ω_0 as a Banach space, such as separability, duality, reflexivity, uniform, convergence etc.

Theorem 1 — $(\Omega_0, \| \,.\, \|)$ is non-uniformly convex Banach space which is separable also.

PROOF : Let $\{f_p\}$ be a Cauchy sequence in Ω_0 such that $f_p(s) = \sum_{n=1}^{\infty} a_{pn} e^{s\lambda_n}$, then $\left| \frac{a_{pn}}{\alpha_n} \right| \to 0$ for each p and $n \to \infty$. As f_p is Cauchy sequence so for a given $\epsilon > 0$, there exists a positive integer N_0 such that

$$\| f_p - f_q \| \leqslant \epsilon \quad \text{for} \quad p, q \geqslant N_0(\epsilon)$$

$$\Rightarrow \sup_n \left| \frac{a_{pn} - a_{qn}}{\alpha_n} \right| \leqslant \epsilon \quad \text{for} \quad p, q \geqslant N_0(\epsilon)$$

$$\Rightarrow \left| \frac{a_{pn} - a_{qn}}{\alpha_n} \right| \leqslant \epsilon \quad \text{for each } n \text{ and } p, q \geqslant N_0(\epsilon). \qquad \text{...(1.4)}$$

This shows that $\left\{ \frac{a_{pn}}{\alpha_n} \right\}$ is a Cauchy sequence in C for each n, so converges to $\frac{a_{on}}{\alpha_n}$ (say) as $p \to \infty$. Further for $n \to \infty$

$$\left|\frac{a_{0n}}{\alpha_n}\right| = \left|\frac{a_{0n}}{\alpha_n} - \frac{a_{pn}}{\alpha_n} + \frac{a_{pn}}{\alpha_n}\right| \leqslant \left|\frac{a_{0n}}{\alpha_n} - \frac{a_{pn}}{\alpha_n}\right| + \left|\frac{a_{pn}}{\alpha_n}\right| \to 0$$

hence f, where $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ belongs to Ω_0. Further

$$\| f_p - f \| = \sup_n \left|\frac{a_{pn} - a_{0n}}{\alpha_n}\right| \to 0 \quad \text{as} \quad p \to \infty \quad \text{[by (1.4)].}$$

Thus Ω_0 is complete and therefore a Banach space. Further consider f and g defined as follows :

$$f(s) = \alpha_{m_0} e^{s\lambda_{m_0}} \quad \text{and} \quad g(s) = \alpha_{m_0} e^{s\lambda_{m_0}} + \alpha_k e^{s\lambda_k}$$

where m_0 and k are fixed positive integers. Obviously $f, g \in \Omega_0$ and

$$\| f \| = \| g \| = 1, \ \| f - g \| = 1 > \epsilon,$$

but $\| f + g \| = 2 \nleq 2 - 2\delta$, for any positive $\delta(\epsilon)$ showing that Ω_0 is not uniformly convex (Wilnasky 1969).

It is yet to be shown that Ω_0 is separable. For it, consider the set of all function f, in Ω_0 which has the representation $f(s) = \sum_{n=1}^{m} b_n e^{s\lambda_n}$ where m is a finite positive integer and $b_n = r_n + it_n$ is such that r_n and t_n are rational nos. for each n. This set is readily seen to be a countable one. It is also every where dense in Ω_0, for let $\epsilon > 0$ be given and $f \in \Omega_0$, $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, then $\left|\frac{a_n}{\alpha_n}\right| \to 0 \ (n \to \infty)$ which implies that $\left|\frac{a_n}{\alpha_n}\right| < \epsilon/2$ for $n > N_0$, N_0 being suitably chosen positive integer which in turn implies $\sup_{n>N_0} \left|\frac{a_n}{\alpha_n}\right| \leqslant \epsilon/2$.

Let $g \in \Omega_0$ be defined as $g(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$ where b_n's are given as

$$b_n = 0 \quad \text{for} \quad n > N_0 \quad \text{and}$$

$$\left|\frac{a_n - b_n}{\alpha_n}\right| < \tfrac{1}{2}\epsilon \quad \text{for} \quad n = 1, 2, 3 \ldots N_0$$

then $\| f - g \| \leqslant \sup_{n \leqslant N_0} \left|\frac{a_n - b_n}{\alpha_n}\right| + \sup_{n>N_0} \left|\frac{a_n}{\alpha_n}\right| \leqslant \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$

Hence Ω_0 is separable and this completes the proof of the theorem.

2. Uniform Convergence

In this section we study uniform convergence etc.

Theorem 2 — $f_p \to f$ in $\Omega_0 \Rightarrow f_p(s) \to f(s)$ uniformly over compact subsets of R where

$$f_p(s) = \sum_{n=1}^{\infty} a_{pn} e^{s\lambda_n} \quad \text{and} \quad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}.$$

Proof: Let S be a compact subset in the region of convergence R, then we can get a rectangle $T = \left\{(\sigma, t) \middle| \begin{array}{l} \sigma_1 \leqslant \sigma \leqslant \sigma_2 \\ t_1 \leqslant t \leqslant t_2 \end{array}\right\}$ in R containing S. Let σ_3 satisfy the condition $\sigma_2 < \sigma_3 < \alpha$ so that $\theta = \dfrac{e^{\sigma_2}}{e^{\sigma_3}} < 1$. Then for a given $\epsilon > 0$, choose η s.t.

$$\eta K \sum_{n=1}^{\infty} \theta^{\lambda_n} < \epsilon \quad \text{where} \quad |\alpha_n| e^{\sigma_3 \lambda_n} \leqslant K < \infty \quad \text{and}$$

$$\| f_p - f \| \leqslant \eta \quad \text{for} \quad p \geqslant p_0(\eta)$$

$$\Rightarrow \sup_n \left| \frac{a_{pn} - a_n}{\alpha_n} \right| \leqslant \eta \quad \text{for} \quad p \geqslant p_0(\eta)$$

or $$\left| \frac{a_{pn} - a_n}{\alpha_n} \right| \leqslant \eta \quad \text{for each } n \text{ and} \quad p \geqslant p_0(\eta)$$

then for s in S, $p \geqslant p_0$

$$| f_p(s) - f(s) | \leqslant \sum_{n=1}^{\infty} | a_{pn} - a_n | e^{\sigma\lambda_n} = \sum_{n=1}^{\infty} \left| \frac{a_{pn} - a_n}{\alpha_n} \right| | \alpha_n | e^{\sigma\lambda_n}$$

$$\leqslant \eta \sum_{n=1}^{\infty} | \alpha_n | e^{\sigma_2 \lambda_n} \leqslant \eta K \sum_{n=1}^{\infty} \theta^{\lambda_n} < \epsilon.$$

Theorem 3 — Every bounded linear functional 'L' defined for $f \in \Omega_0$ is of the form

$$L(f) = \sum_{n=1}^{\infty} a_n t_n, \; f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

where $$\sum_{n=1}^{\infty} | \alpha_n t_n | < \infty.$$

To prove the theorem we require the following lemma.

Lemma — $f_m \to f (m \to \infty)$ where $f_m(s) = \sum_{n=1}^{m} a_n e^{s\lambda_n}$ and $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ for s belonging to the region of convergence R, iff $\left|\dfrac{a_n}{\alpha_n}\right| \to 0 \; (n \to \infty)$ i.e. $f \in \Omega_0$.

PROOF : If $f \in \Omega_0$ where $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ we have $\left|\frac{a_n}{\alpha_n}\right| \to 0 \ (n \to \infty)$ and

$\| f - f_m \| = \sup_{n>m} \left|\frac{a_n}{\alpha_n}\right|$ which goes to zero as $m \to \infty$. Conversely if $f \notin \Omega_0$,

$\| f_{pq} \| = \max_{p \leqslant n \leqslant q} \left|\frac{a_n}{\alpha_n}\right|$ where $f_{pq}(s) = \sum_{n=p}^{q} a_n e^{s\lambda_n}$ so that $\{f_m\}$ is not even a Cauchy's sequence.

PROOF OF THE THEOREM : Let L be defined on f in Ω_0 as follows :

$$L(f) = \sum_{n=1}^{\infty} a_n t_n \quad \text{where} \quad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}.$$

Since $\sum_{n=1}^{\infty} | a_n t_n | \leqslant \sup_n \left|\frac{a_n}{\alpha_n}\right| \sum_{n=1}^{\infty} | \alpha_n t_n | = \| f \| \sum_{n=1}^{\infty} | \alpha_n t_n | < \infty$

hence L is well defined functional on Ω_0 further

$$| L(f) | \leqslant \Sigma | a_n t_n | \leqslant \| f \| \Sigma | \alpha_n t_n | , \text{ this gives}$$

$$\| L \| \leqslant \Sigma | \alpha_n t_n | , \qquad \ldots(2.1)$$

therefore L is bounded linear functional on Ω_0 so it is an element of Ω_0^*, the dual space of Ω_0. Conversely if $L \in \Omega_0^*$ and be defined as $L(\delta_n) = t_n$ where $\delta_n \in \Omega_0$ is given as $\delta_n(s) = e^{s\lambda_n}$ for each n, then for any $f \in \Omega_0$

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} = \sum_{n=1}^{\infty} a_n \delta_n(s)$$

$$L(F) = L(\lim_{m\to\infty} f_m) = L(\lim_{m\to\infty} \sum_{n=1}^{m} a_n \delta_n) = \lim_{m\to\infty} [\sum_{n=1}^{m} a_n L(\delta_n)]$$

$$= \sum_{n=1}^{\infty} a_n t_n$$

Now we show that $\Sigma | \alpha_n t_n | \leqslant \| L \|$ so that $\Sigma | \alpha_n t_n | < \infty$. For it take any $r \geqslant 1$. and let

$$a_n = \begin{cases} | \alpha_n | \operatorname{sgn}(t_n) & \text{for } 1 \leqslant n \leqslant r \\ 0 & \text{for } n > r \end{cases}$$

Now define f, as $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, then obviously $f \in \Omega_0$ and $\| f \| = 1$ hence

$$| L(f) | = | \sum_{n=1}^{r} | \alpha_n | \operatorname{sgn}(t_n) L(\delta_n) | = \sum_{n=1}^{r} | \alpha_n t_n | \text{ but}$$

$$| L(f) | \leqslant \| L \| \cdot \| f \| = \| L \|$$

so that $\sum_{n=1}^{r} | \alpha_n t_n | \leqslant \| L \|$ and

$$\sum_{n=1}^{\infty} | \alpha_n t_n | = \sup_r \sum_{n=1}^{r} | \alpha_n t_n | \leqslant \sup_r \| L \| = \| L \|. \qquad \ldots(2.2)$$

(2.1) and (2.2) together show that $\sum_{n=1}^{\infty} | \alpha_n t_n | = \| L \|$ and this completes the proof of the theorem.

3. Characterization of Bounded Linear Transformation

Let $(\xi_{nk})_{n,k \in N}$ be an infinite matrix of complex entries. Let A be a transformation on Ω_0 defined as $(A(f)(s)) = \sum_{n=1}^{\infty} A_n(f) e^{s\lambda_n}$ where $A_n(f) = \sum_{k=1}^{\infty} \xi_{n_k a_k}$ and f belonging to Ω_0 is given as $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$. We have already shown in Juneja and Srivastava (1979) that for $A(f)$ to be meaningful $\sum_{k=1}^{\infty} | \xi_{n_k \alpha_k} |$ should converge for each n.

Theorem 4 — Let

(i) $\frac{\xi_{nk}}{\alpha_n} \to 0 \quad (n \to \infty, k \text{ fixed})$ and

(ii) $M = \sup_n \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}}{\alpha_n} \alpha_k \right| < \infty$ then A defined as above is a bounded linear operator on Ω_0 such that $\| A \| = M$.

Proof : Let $f \in \Omega_0$ then $A(f)$ defined as above will belong to Ω_0 provided $\left| \frac{A_n(f)}{\alpha_n} \right| \to 0 \ (n \to \infty)$. For it

$$\left| \frac{A_n(f)}{\alpha_n} \right| = \sum_{k=1}^{m} \left| \frac{\xi_{nk} a_k}{\alpha_n} \right| + \sum_{k=m+1}^{\infty} \left| \frac{\xi_{nk} a_k}{\alpha_n} \right|$$

(*equation continued on p.* 527)

$$< \| f \| \sum_{k=1}^{m} \left| \frac{\xi_{nk}\alpha_k}{\alpha_n} \right| + \left(\max_{k \geqslant m+1} \left| \frac{a_k}{\alpha_k} \right| \right) M$$

Now we can choose 'm' so large that $\max_{k \geqslant m+1} \left| \frac{a_k}{\alpha_k} \right| < \epsilon$, for a given $\epsilon > 0$ and then we take 'n' so large that $\sum_{k=1}^{m} \left| \frac{\xi_{nk}}{\alpha_n} \alpha_k \right| < \epsilon$ {It is possible, since $\xi_{nk}/\alpha_n \to 0$ ($n \to \infty$, k fixed) by (1) which implies $\frac{\xi_{nk}}{\alpha_n} \alpha_k \to 0$ ($n \to \infty$, k fixed)} and therefore, in this way we can make $\left| \frac{A_n(f)}{\alpha_n} \right|$ as small as we please by suitable choice of 'm' and 'n', hence $\left| \frac{A_n(f)}{\alpha_n} \right| \to 0$ ($n \to \infty$).

A is linear, this much is trivial. Further

$$\| A(f) \| \leqslant \| f \| \sup_{n} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}}{\alpha_n} \alpha_k \right| = M \| f \| \qquad \text{...(3.1)}$$

showing that A is bounded and

$$\| A \| \leqslant M. \qquad \text{...(3.2)}$$

Further if $\epsilon > 0$ be given then there exists a positive integer $m = m(\epsilon)$ such that the given condition (3.1) implies

$$\sum_{k=1}^{\infty} \left| \frac{\xi_{mk}}{\alpha_m} \alpha_k \right| > M - \tfrac{1}{2} \epsilon. \text{ But } \sum_{k=1}^{\infty} \left| \frac{\xi_{mk}}{\alpha_m} \alpha_k \right|$$

is finite so there exist a positive integer $p = p(\epsilon)$ such that

$$\sum_{k>p} \left| \frac{\xi_{mk}}{\alpha_m} \alpha_k \right| < \tfrac{1}{2} \epsilon. \quad \text{Now let}$$

$$a_k = \begin{cases} \alpha_k \operatorname{sgn} \xi_{mk} & \text{for } 1 \leqslant k \leqslant p \\ 0 & k > p \end{cases}$$

then f, given as $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ is such that $f \in \Omega_0$ and $\| f \| = 1$. For this f, we have

$$\| A(f) \| / \| f \| = \| A(f) \| = \sup_{n} \left| \frac{A_n(f)}{\alpha_n} \right| \geqslant \left| \frac{A_m(f)}{\alpha_m} \right| > M - \epsilon$$

but $\| A \| = \sup \frac{\| A(f) \|}{\| f \|} : f \neq \theta$ where $\theta(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ is s.t. $a_n = 0$ for each n and therefore

$$\| A \| \geqslant M. \qquad \ldots(3.3)$$

By (3.2) and (3.3) we get $\| A \| = M$ and this completes the proof of the theorem.

The following theorem is the converse of this theorem.

Theorem 5 — If A be a bounded linear transformation on Ω_0, then A determines a matrix (ξ_{nk}), $(n, k = 1, 2, 3 \ldots)$ such that $A_n(f) = \sum_{k=1}^{\infty} \xi_{nk} a_k$ and conditions (i) and (ii) of Theorem 4 hold where $(A(f)(s)) = \sum_{n=1}^{\infty} A_n(f) e^{s\lambda_n}$, $f \in \Omega_0$ is given as $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$.

PROOF: Let A be defined on the set $\{\delta_k, \delta_k(s) = e^{s\lambda_k}, k = 1, 2, 3 \ldots\}$ as $A(\delta_k(s)) = \sum_{n=1}^{\infty} \xi_{nk} e^{s\lambda_n}$ then linearity and boundedness of A yields

$$A(f(s)) = \sum_{k=1}^{\infty} a_k A(\delta_k(s)) = \sum_{k=1}^{\infty} a_k \Big(\sum_{n=1}^{\infty} \xi_{nk} e^{s\lambda_n}\Big)$$

$$= \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{\infty} \xi_{nk} a_k\Big) e^{s\lambda_n} = \sum_{n=1}^{\infty} A_n(f) e^{s\lambda_n}$$

where

$$A_n(f) = \sum_{k=1}^{\infty} \xi_{nk} a_k \quad (n = 1, 2, 3 \ldots)$$

and therefore

$$\frac{A_n(f)}{\alpha_n} = \sum_{k=1}^{\infty} \frac{\xi_{nk}}{\alpha_n} a_k \qquad \ldots(3.4)$$

since $A(\delta_k) \in \Omega_0$ for each k implies $\frac{\xi_{nk}}{\alpha_n} \to 0$ $(n \to \infty)$ for each k. Now it remains to show that $\| A \| = M$. For it we must only show that, there exist H such that $M = \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}}{\alpha_n} \alpha_k \right| \leqslant H$ for each n. Since $\left| \frac{A_n(f)}{\alpha_n} \right| \leqslant \| A(f) \| \leqslant \| A \| \cdot \| f \|$ so that

$\left\{\frac{A_n}{\alpha_n}\right\}$ is a sequence of bounded linear functionals on Ω_0 such that $\lim_{n\to\infty} \frac{A_n(f)}{\alpha_n} = 0$ on Ω_0. Therefore by Banach-Steinhaus theorem it follows that the sequence $\left\{\left\|\frac{A_n}{\alpha_n}\right\|\right\}$ of norms is bounded i.e. $\left\|\frac{A_n}{\alpha_n}\right\| \leqslant H$ where H is some constant for each n. But $\left\|\frac{A_n}{\alpha_n}\right\|$, where $\frac{A_n}{\alpha_n}$ is given by (3.4) by the nature of Ω_0^* is given as

$$\left\|\frac{A_n}{\alpha_n}\right\| = \sum_{k=1}^{\infty} \left|\frac{\xi_{nk}}{\alpha_n}\alpha_k\right| < \infty$$

hence we have shown that

$$M = \sup_n \left(\sum_{k=1}^{\infty} \left|\frac{\xi_{nk}}{\alpha_n}\alpha_k\right|\right) < \infty.$$ Now the proof of Theorem 4 gives $\| A \| = M$.

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