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FLOW VARIATIONS BEHIND A CURVED SHOCK IN AN IDEAL DISSOCIATING GAS

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In this paper, the various intrinsic properties of the steady flow of an ideal dissociating gas behind a curved shock wave are studied. The main objectives of the paper are: (1) to determine the variation of pressure, density, and temperature along stream lines, their principal normals and binormals in terms of the geometry of stream lines and flow field parameters, (2) to determine the vorticity in terms of the stream line geometry and the variations of fluid velocity along the principal normals and binormals, (3) to calculate the variations of flow parameters along the normal to the shock surface, and lastly (4) to determine the curvature of a stream line, the variations of flow parameters along stream lines, their principal normals and binormals at the rear of the shock surface in terms of the shock geometry and flow conditions just ahead of the shock wave.

1. INTRODUCTION

Kanwal (1957) discussed the variations of flow quantities along stream lines, their principal normals and binormals in three dimensional gas flows. Recently, the problem of determining the differential effects of shock fronts on the flow quantities at the rear of the shock has drawn the attention of several authors (Kanwal 1960, Pant and Mishra 1965, Taub 1953, Rishi Ram 1968). Sedney and Gerber (1967) determined the shock curvature and the flow variable gradients at the tip of a pointed body in a non-equilibrium flow. In the present paper, we shall study the various intrinsic properties of the steady flow of an ideal dissociating gas behind a curved shock wave. The comma followed by an index in the following equations denotes partial differentiation. An index which occurs twice in a term is to be summed over the admissible values of the index. Since there is no distinction between covariant and contra-variant indices within a rectangular system, we may write an index as subscript or as superscript without modification of the term in which the index occurs.

2. BASIC EQUATIONS OF THE SYSTEM

The basic equations governing the steady flow, when viscosity and heat conduction are neglected, are (Clarke 1960)

$$\rho_{,i} u_i + \rho u_{i,i} = 0 \qquad \ldots(2.1)$$

$$u_i C_{,i} = \pi \qquad \ldots(2.2)$$

$$P_{,i} + \rho u_j u_{i,j} = 0 \qquad \ldots(2.3)$$

$$u_i P_{,i} + \rho a_f^2 u_{i,i} + \rho a_f^2 \sigma\pi = 0 \qquad \ldots(2.4)$$

where

$$\sigma = \rho \frac{\beta_f}{C_{Pf}} \left(\frac{\partial h}{\partial C}\right)_{P,T} + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial C}\right)_{P,T}$$

$$\pi = \frac{1}{\tau}\{\bar{K}(1-C) - C^2\}$$

$$\beta_f = -\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_{P,C}, \quad C_{Pf} = \left(\frac{\partial h}{\partial T}\right)_{P,C}$$

$$a_f^2 = \frac{4+C}{3}\,\frac{P}{\rho}.$$

The thermal and caloric equation of state for the ideal dissociating gas are given by (Li 1961)

$$P = \rho(1+C)\,RT \qquad \ldots(2.5)$$

$$h = (4+C)\,RT + CD \qquad \ldots(2.6)$$

respectively, where h is the specific enthalpy, and D the dissociation energy per unit mass. σ can be evaluated from the relations (2.5) and (2.6).

Let λ_i, μ_i and ν_i be respectively the components of the unit vectors along the tangent, principal normal and binormal of a stream line. The curvature of a stream line can be expressed in terms of velocity gradients in the form:

$$K\mu_i = \frac{1}{U^2}\left\{u_j u_{i,j} - \frac{u_l u_j u_{l,j}\lambda_i}{U}\right\}. \qquad \ldots(2.7)$$

Using equations (2.2), (2.3) and (2.4) in eqn. (2.7) and multiplying by λ_i, μ_i and ν_i in turn we get

$$\frac{1}{\rho U^2} P_{,i}\lambda_i = \frac{1}{\rho U^2}\frac{dP}{dS} = -\frac{a_f^2}{U^3}[u_{l,l} + \sigma u_l C_{,l}] \qquad \ldots(2.8)$$

$$P_{,i}\mu_i = \frac{dP}{dn} = -K\rho U^2 \qquad \ldots(2.9)$$

$$P_{,i}\nu_i = \frac{dP}{db} = 0 \qquad \ldots(2.10)$$

Equations (2.8), (2.9) and (2.10) give variations in pressure along stream lines, their principal normals and binormals. It is interesting to note that the pressure remains constant along a binormal of the stream line at each point.

If (K', K'') and (σ', σ'') be the curvatures and torsions of the curves of congruences determined by the principal normals and binormals of the streamlines, the following geometric relations hold:

$$\frac{d\lambda_i}{dS} = K\mu_i$$

$$\frac{d\lambda_i}{dn} = -K'\mu_i + \sigma'\nu_i$$

$$\frac{d\lambda_i}{db} = \sigma''\mu_i - K''\nu_i$$

from which we get

$$\lambda_{i,j} = K\mu_i\lambda_j + (\sigma'\nu_i - K'\mu_i)\,\mu_j + (\sigma''\mu_i - K''\nu_i)\,\nu_j. \qquad \text{...(2.11)}$$

We can also write $\lambda_{i,i}$ in the form

$$\lambda_{i,i} = \left\{\frac{u_{i,i}}{U} - \frac{u_i}{U^3}\,u_i u_{i,i}\right\}. \qquad \text{...(2.12)}$$

From (2.11) and (2.12) we get

$$u_{i,i} = \frac{M^2 U}{M^2 - 1}\left\{\frac{\sigma\pi}{UM^2} - (K' + K'')\right\} \qquad \text{...(2.13)}$$

where $\quad M = U/a_f$.

Substituting from (2.13) in (2.8) we get

$$\frac{dP}{dS} = \frac{\rho U}{M^2 - 1}\{-\sigma\pi + U(K' + K'')\}. \qquad \text{...(2.14)}$$

Making use of the relation (2.13) in eqn. (2.1) we get

$$\frac{d\rho}{dS} = \frac{\rho}{(M^2 - 1)\,U}\{-\sigma\pi + M^2 U(K' + K'')\}. \qquad \text{...(2.15)}$$

Using (2.2), (2.14) and (2.15) in relation (2.6) we get

$$\frac{dT}{dS} = \frac{1}{(1 + C)\,R(M^2 - 1)}\left\{(K' + K'')\left(U^2 - \frac{P}{\rho}\,M^2\right) + \frac{\sigma\pi}{U}\left(\frac{P}{\rho} - U^2\right)\right\} - \frac{P}{\rho R(1 + C)^2}\,\frac{\pi}{U}. \qquad \text{...(2.16)}$$

Equations (2.14), (2.15) and (2.16) determine the variation of pressure, density and temperature along streamlines.

3. Determination of Vorticity

Even if the flow ahead of a shock wave is irrotational, the flow just behind the shock surface is in general rotational (Rishi Ram 1968).

The components ω_i of vorticity are given by $\omega_i = e_{ijk}u_{k,j}$ which can be written as

$$\omega_i = Ue_{ijk}\lambda_{k,j} + \frac{1}{U} e_{ijk}\lambda_k u_l u_{l,j} \quad \text{...(3.1)}$$

Multiplying (3.1) by λ_i, μ_i and ν_i in turn and making use of (2.11) we get

$$\omega_i\lambda_i = U(\sigma' - \sigma'') \quad \text{...(3.2)}$$

$$\omega_i\mu_i = \frac{dU}{db} \quad \text{...(3.3)}$$

$$\omega_i\nu_i = UK - \frac{dU}{dn} \quad \text{...(3.4)}$$

Equations (3.2), (3.3) and (3.4) determine the vorticity components along a streamline and its principal normal and binormal. By virtue of (3.2), (3.3) and (3.4) we have

$$\omega_i = U(\sigma' - \sigma'')\lambda_i + \frac{dU}{db}\mu_i + \left(UK - \frac{dU}{dn}\right)\nu_i. \quad \text{...(3.5)}$$

Equation (3.5) determines the vorticity components in terms of the streamline geometry and the fluid velocity variations along the principal normal and binormal of the stream line.

4. Variations Along the Shock Surface

If n_i and $x^i_{,\alpha}$ are respectively the components of the unit normal and projective tensor of the shock surface, we define $u_{n/} = u_i n_i$ and $u_\alpha = x^i_{,\alpha} u_i$. To determine the variations of flow variables along the shock surface, we assume the shape of the shock to be known and the flow ahead to be uniform. The jump condition expressing the values of flow variables just behind the shock surface in terms of those just ahead are given by (Thomas 1969)

$$[u_i] = \frac{-\delta}{(1+\delta)} u_{1n/} n_i;\ [P] = \frac{\delta\rho_1 u^2_{1n/}}{(1+\delta)};\ [\rho] = \delta\rho_1 \quad \text{...(4.1)}$$

where δ is the density strength of the shock wave which can be determined from the following shock condition due to the conservation of energy across the shock (Thomas 1969, Li 1961) :

$$\rho u_{1n/}\left[\left(\frac{4+C}{1+C}\right)\frac{P}{\rho} + \alpha D + \tfrac{1}{2}U^2\right] = 0. \quad \text{...(4.2)}$$

From the Weingarten's formula, we can obtain the relation

$$u_{1n/,\alpha} = -\, g^{\theta\beta} b_{\beta\alpha} u_{1\theta} \quad \text{...(4.3)}$$

where $g^{\theta\beta}$ and $b_{\beta\alpha}$ are respectively the components of contravariant first fundamental tensor and covariant second fundamental tensor of the shock surface and u_θ are the components of gas velocity along the shock surface.

By virtue of eqn. (4.3) the variations of flow variables along the shock surface can be obtained by differentiating the jump conditions (4.1). Thus the variations of flow variables along the shock surface just behind it can be expressed in terms of the surface geometry of the shock and the flow parameters ahead of it in the following forms:

$$\rho_{,\alpha} = \rho_1 \delta_{,\alpha}$$

$$u_{i,\alpha} = \frac{\delta u_{1n/}\, g^{\theta\beta} b_{\beta\alpha} x^i_{,\theta}}{(1+\delta)} + \frac{\delta n_i}{(1+\delta)} g^{\theta\phi} b_{\phi\alpha} u_{1\theta} - \frac{u_{1n/}}{(1+\delta)^2} n_i \delta_{,\alpha} \qquad ...(4.4)$$

$$P_{,\alpha} = \frac{\rho_1 u^2_{1n/}\, \delta_{,\alpha}}{(1+\delta)^2} - \frac{2\delta \rho_1 u_{1n/}\, g^{\beta\theta} b_{\beta\alpha} u_{1\theta}}{(1+\delta)}. \qquad ...(4.5)$$

5. Variations Along the Normal Trajectory of the Shock Surface

The flow ahead of a shock wave is assumed to be uniform. As a result of the passage of a shock wave, the medium undergoes abrupt changes in flow and field parameters. Let $[z] = z_0 - z_1$ denote the jump in the quantity enclosed across a shock wave, where z_0 and z_1 are values of z just behind and just ahead of a shock wave. Let $\partial_n f$ and $f_{,\alpha}$ denote the derivative of f along the normal and tangent to the shock surface respectively. At any point of the shock surface we have a geometrical relation

$$n_i n_j = \delta^j_i - g^{\alpha\beta} x^i_{,\alpha}\, x^j_{,\beta} \qquad ...(5.1)$$

so that the jump in the gradient of any flow parameter z can be expressed in the form

$$[z_{,i}] = n_i \partial_n z + x^i_{,\beta}\, g^{\alpha\beta} z_{,\alpha}. \qquad ...(5.2)$$

By virtue of the eqn. (5.2) the eqns. (2.1) to (2.4) can be transformed into the following forms:

$$\rho n_i \partial_n u_i + u_{n/}\, \partial_n \rho + \rho u_{i,\alpha} x^i_{,\beta}\, g^{\alpha\beta} + u^\alpha \rho_{,\alpha} = 0 \qquad ...(5.3)$$

$$u_{n/}\, \partial_n C + u^\alpha C_{,\alpha} = \pi \qquad ...(5.4)$$

$$n_i \partial_n P + \rho u_{n/}\, \partial_n u_i + \rho u^\alpha u_{i,\alpha} + x^i_{,\beta}\, g^{\alpha\beta} P_{,\alpha} = 0 \qquad ...(5.5)$$

$$u_{n/}\, \partial_n P + \rho a_f^2\, n_i \partial_n u_i + \rho a_f^2\, \sigma\pi + u^\alpha P_{,\alpha} + \rho a_f^2\, x^i_{,\beta}\, g^{\alpha\beta} u_{i,\alpha} = 0 \qquad ...(5.6)$$

where $u_{i,\alpha}$, $P_{,\alpha}$ and $\rho_{,\alpha}$ are determined from eqns. (4.4) and (4.5).

Multiplying eqn. (5.5) by n_i, and solving for $\partial_n P$ and $n_i \partial_n u_i$ with the help of (5.5) and (5.6) we get

$$\partial_n P = \frac{1}{(a_f^2 - u_{n/}^2)} [(\rho a_f^2 \, \sigma\pi + u^\alpha P_{,\alpha} + \rho a_f^2 \, x^i_{,\beta} \, g^{\alpha\beta} u_{i,\alpha}) \, u_{n/} - a_f^2 \, \rho n_i u^\alpha u_{i,\alpha}] \quad \ldots(5.7)$$

$$n_i \partial_n u_i = \frac{-1}{\rho u_{n/} (a_f^2 - u_{n/}^2)} [u_{n/} \, (\rho a_f^2 \, \sigma\pi + \rho a_f^2 \, x^i_{,\beta} \, g^{\alpha\beta} u_{i,\alpha} + u^\alpha P_{,\alpha}) - \rho a_f^2 \, n_i u^\alpha u_{i,\alpha}] - n_i u^\alpha u_{i,\alpha} \quad \ldots(5.8)$$

$$u_{n/} \frac{\partial \rho}{\partial n} = \frac{1}{(a_f^2 - u_{n/}^2)} [\rho a_f^2 \, \sigma\pi + u^\alpha P_{,\alpha} + \rho a_f^2 \, x^i_{,\beta} \, g^{\alpha\beta} u_{i,\alpha} - \rho u_{n/} \, n_i u^\alpha u_{i,\alpha}] - \rho u_{i,\alpha} x^i_{,\beta} \, g^{\alpha\beta} - u^\alpha \rho_{,\alpha} \quad \ldots(5.9)$$

which determine the variations of velocity, pressure and density along the normal trajectory of the shock surface.

6. Curvature of a Stream Line at the Rear of Shock Surface

Using (5.2) in (2.7) we obtain

$$K^2 = \frac{1}{U^4} \left\{ u_{n/}^2 \, A_i A_i + u^\alpha u^\alpha u_{i,\alpha} u_{i,\alpha} + 2 u_{n/} \, u^\alpha A_i u_{i,\alpha} - \frac{1}{U^2} \, u_i u_j A_i A_j \right\} \quad \ldots(6.1)$$

where $A_i = \partial_n u_i$. The values of A_i and $u_{i,\alpha}$ are known from (5.8) and (4.4). Hence eqn. (6.1) determines the curvature of a stream line at the rear of a shock surface.

The components μ_i and ν_i of a principal normal and binormal to stream line at the rear of shock surface can also be determined by using the geometrical relation

$$\nu_i = e_{ijk} \lambda_j \mu_k. \quad \ldots(6.2)$$

Using (5.2) in (2.7) we get

$$\mu_i = \frac{1}{KU^2} \left\{ u_{n/} \, A_i + u^\alpha u_{i,\alpha} - \frac{1}{U^2} (u_{n/} \, u_i u_j A_j + u^\alpha u_i u_j u_{j,\alpha}) \right\} \quad \ldots(6.3)$$

and

$$\nu_i = \frac{1}{KU^3} \{ u_{n/} \, A_k + u^\alpha u_{k,\alpha} \} \, e_{ijk} u_j. \quad \ldots(6.4)$$

The gradients of flow variables at the rear of the shock surface can be written as

$$u_{i,j} = \partial_n u_i n_j + g^{\alpha\beta} u_{i,\alpha} x^j_{,\beta} \qquad \ldots(6.5)$$

$$P_{,i} = \partial_n P n_i + g^{\alpha\beta} P_{,\alpha} x^i_{,\beta} \qquad \ldots(6.6)$$

$$\rho_{,i} = \partial_n \rho n_i + g^{\alpha\beta} \rho_{,\alpha} x^i_{,\beta} \, . \qquad \ldots(6.7)$$

Multiplying eqns. (6.5) to (6.7) by λ_i, μ_i and ν_i in turn and using eqns. (5.7) to (6.4) we can determine the variations of flow parameters along stream lines, their principal normals and binormals at the rear of the shock surface in terms of the shock geometry and flow conditions just ahead of the shock wave.

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