Indian J. pure appl. Math., 12(4) : 537–543, April 1981

COMMENTS ON 'BEHAVIOUR OF VISCOELASTIC LUBRICANTS IN REFERENCE TO HUMAN JOINTS'

PEEYUSH CHANDRA

The Mehta Research Institute of Mathematics and Mathematical Physics, 26, Dilkusha, New Katra, Allahabad 211002 (*U.P.*)

(*Received* 3 *July* 1980; *after revision* 30 *October* 1980)

The purpose of this paper is to present corrected analysis of squeeze film lubrication, as considered by Tandon and Agarwal (1979), with viscoelastic fluid as lubricant between two plates, one of which is porous. Approximate analytical expression for load carrying capacity (W) has been obtained and numerical results showing the effect of various parameters on W have been presented.

INTRODUCTION

Tandon and Agarwal (1979) considered the problem of 'squeeze film lubrication with viscoelastic fluid as the lubricant in between two approaching surfaces one of which is covered with porous material'. They obtained the expressions for pressure and load capacity in the form of perturbation series with the perturbation parameter ϵ,

$$\epsilon = (\lambda_1 - \lambda_2)\, \mu_0/T^2 \qquad ...(1)$$

where λ_1, λ_2, μ_0 are viscoelastic constants with dimensions of time and T is some unit time (Mow 1968). However, while carrying out their analysis, they have made following mistakes which make the analysis incorrect:

(R1) While solving eqn. (2) for pressure $\bar{p}(x, y)$ in the porous region $-H \leqslant y \leqslant 0$ and $-L \leqslant x \leqslant L$ they have obtained

$$\bar{p} = \Sigma A_n \{\exp \gamma_n y + \exp(-\gamma_n(2H + y))\} \sin \gamma_n x$$

which gives $\bar{p}(x, y) = -\bar{p}(-x, y)$ and hence $p(x) = -p(-x)$. This is not correct from physical point of view and would give

$$W = \int_{-L}^{L} p\, dx = 0$$

instead of their expression for calculating W i.e.

$$W = 2 \int_0^L p\, dx.$$

(R2) They have wrongly separated the boundary condition (9b) i.e.

$$v = -\frac{k}{\eta_0}\frac{\partial \bar{p}}{\partial y} \quad \text{at} \quad y = 0$$

for various orders in ϵ.

(R3) They have plotted the normalized load carrying capacity $\bar{W}$ vs. M where,

$$\bar{W} = (\bar{W}_0 + \epsilon\bar{W}_1)/\bar{W}_0$$

$$= 1 + (\epsilon\bar{W}_1/\bar{W}_0)$$

and have related their results to the study of human joint load capacity. This is misleading as their normalizing factor $\bar{W}_0$ itself depends upon M and therefore behaviour of $\bar{W}$ w.r.t. M cannot be expected to give the behaviour of W versus M. Moreover, they have not taken any values for the parameter T.

In view of the above three remarks, it is required to present the correct analysis which is the purpose of this article.

Analysis

As in the paper of Tandon and Agarwal, we consider here the squeeze film behaviour of viscoelastic fluid between two plates when one plate (say at $y = 0$) has porous facing and the other (i.e. at $y = h$) is moving normally downwards with squeeze velocity $V(= -dh/dt)$. The porous facing is assumed to have uniform thickness H and permeability k. However, in the light of ($R1$), we shall have the co-ordinate system at one end of the lower plate; so if the plate has breadth b and length L, then $0 \leqslant x \leqslant L$ instead of $-L \leqslant x \leqslant L$ as taken by Tandon and Agarwal (1979).

Now, while considering the flow in porous region, we assume that the pressure $\bar{p}(x, y)$ in this region would be governed by Laplace equation [Wu 1972, eqn. (2) of Tandon and Agarwal (1979)], i.e.

$$\frac{\partial^2 \bar{p}}{\partial x^2} + \frac{\partial^2 \bar{p}}{\partial y^2} = 0. \qquad ...(2)$$

Further, following Mow (1968), the governing equation for fluid film region are given as [eqn. (11) of Tandon and Agarwal (1979)]

$$\frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2}\left[1 - 3(\lambda_1 - \lambda_2)\,\mu_0 \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad ...(3a)$$

$$\frac{\partial p}{\partial y} = 0 \qquad ...(3b)$$

alongwith the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots(4)$$

where, p is the pressure, and u, v are axial and normal velocity components in the fluid film region and η is viscosity of suspension as taken by Tandon and Agarwal (1979) i.e.

$$\eta = \eta_0(1 + 2.5\,\phi). \quad \ldots(5)$$

We assume no slip condition for u at the two surfaces (at $y = 0$ and $y = h$), and for v we have

$$v = \frac{dh}{dt} = -V \quad \text{at} \quad y = h \quad \ldots(6)$$

$$v = -\frac{k}{\eta_0}\frac{\partial \bar{p}}{\partial y} \quad \text{at} \quad y = 0. \quad \ldots(7)$$

The conditions for p and $\bar{p}$ are

$$p(0) = p(L) = 0 = \bar{p}(0, y) = \bar{p}(L, y) \quad \ldots(8)$$

and

$$\frac{\partial \bar{p}}{\partial y}(x, -H) = 0. \quad \ldots(9)$$

Also, we would like to explicitly mention the condition for continuity of pressure at the interface of porous and fluid film regions (Wu 1972, Sparrow *et al.* 1972) i.e.

$$\bar{p}(x, 0) = p(x). \quad \ldots(10)$$

We now assume series form for u, v and p in terms of ϵ [as defined in eqn. (1)],

$$q = q_0 + \epsilon q_1 + \cdots \quad \ldots(11)$$

where q represents u, v and p.

It may be noted that series form for p alongwith the condition of continuity of pressure [i.e. eqn. (10)] suggest series form for $\bar{p}$ also, and so,

$$\bar{p}(x, y) = \bar{p}_0(x, y) + \epsilon \bar{p}_1(x, y) + \ldots . \quad \ldots(12)$$

This has been missed by Tandon and Agarwal (1979) and hence they wrongly separated [in their eqn. (16*c*)] eqn. (7) [their eqn. (9b)] for various orders as

$$\left.\begin{aligned} v_0 &= -\frac{k}{\eta_0}\frac{\partial \bar{p}}{\partial y} \\ v_1 &= 0 = v_2 = \ldots \end{aligned}\right\} \text{ at } y = 0. \qquad \ldots(16c)$$

However, it can now be seen that on substituting expressions for v and $\bar{p}$ in eqns. (6) and (7), we get the following conditions for v_i,

$$\left.\begin{aligned} &v_0 = \dot{h},\ v_1 = 0 = v_2 = \ldots \quad \text{at} \quad y = h \\ \text{and} \quad &v_i = -\frac{k}{\eta_0}\frac{\partial \bar{p}_i}{\partial y}\ (i = 0, 1, \ldots) \ \text{at} \ y = 0. \end{aligned}\right\} \qquad \ldots(13)$$

Substituting appropriate expansions in eqns. (2) and (3) and eliminating u from eqns. (3) and (4), we get the following equations for $\bar{p}_i$ and p_i :

$$\frac{\partial^2 \bar{p}_i}{\partial x^2} + \frac{\partial^2 \bar{p}_i}{\partial y^2} = 0 \quad i = 0, 1, \ldots \qquad \ldots(14)$$

and

$$\frac{h^3}{12\eta}\frac{d^2 p_0}{dx^2} = -V + \frac{k}{\eta_0}\left(\frac{\partial \bar{p}_0}{\partial y}\right)_{y=0} \qquad \ldots(15)$$

$$\frac{h^3}{12\eta}\frac{d^2 p_1}{dx^2} = \frac{k}{\eta_0}\left(\frac{\partial \bar{p}_1}{\partial y}\right)_{y=0} - \frac{3T^2h^5}{80\eta^3}\left(\frac{dp_0}{dx}\right)^2\frac{d^2 p_0}{dx^2} \qquad \ldots(16)$$

and so on.

The boundary conditions will become

$$p_i(0) = p_i(L) = \bar{p}_i(0, y) = \bar{p}_i(L, y) = 0$$

and $$\frac{\partial \bar{p}_i}{\partial y} = 0 \quad \text{at} \quad y = -H.$$

Solving (14) with proper boundary conditions, we get

$$\bar{p}_i = \sum_{n=0}^{\infty} A_n^{(i)} \{\exp(\gamma_n y) + \exp(-\gamma_n(2H + y))\} \sin \gamma_n x \qquad \ldots(17)$$

where $\gamma_n = n\pi/L$, $A_n^{(i)}$ can be determined by using the condition of continuity of pressures in the two regions i.e.

$$p_i(x) = \bar{p}_i(x, 0). \qquad \ldots(18)$$

On substituting the expression for $\bar{p}_0$ from eqn. (17) ($i = 0$) in (15) and solving it with the proper boundary conditions, we get following expression for p_0 (following Sparrow *et al.* 1972).

$$p_0 = \frac{\eta_0 V L^2}{h^3} \sum_{n=1}^{\infty} B_n^{(0)} \sin \gamma_n x \qquad \ldots(19)$$

where

$$\left.\begin{aligned} B_n^{(0)} \,(n \text{ even}) &= 0 \\ B_n^{(0)} \,(n \text{ odd}) &= \frac{48}{n^2\pi^2(n\pi\bar{\eta} + \bar{k}G_n)} \end{aligned}\right\} \qquad \ldots(20)$$

$$G_n = 12(1 - \exp(-2n\pi\bar{H}))/(1 + \exp(-2n\pi\bar{H}))$$

$$\bar{\eta} = \eta_0/\eta = (1 + 2.5\phi)^{-1}, \ \bar{k} = kL/h^3 \quad \text{and} \quad \bar{H} = H/L.$$

Now substituting the expressions for p_0 and $\bar{p}_1$ in (16) and solving, we get

$$p_1 = -\frac{\eta_0 V L^2}{h^3}\left(\frac{3\pi^4\bar{\eta}^3}{320}\,\frac{L^2T^2V^2}{h^4}\right)\sum_{n=1}^{\infty}\frac{n\pi}{2}\,B_n^{(0)}\,B_n^{(1)}\sin\gamma_n x \qquad \ldots(21)$$

where

$$B_n^{(1)}\,(n \text{ even}) = 0$$

$$\begin{aligned} B_n^{(1)}\,(n \text{ odd}) = {} & \sum_{m=1}^{\infty} m^2 B_m^{(0)} \Big[\sum_{r=1}^{\infty} r B_r^{(0)} \{ (|n-m| + r) \\ & \times B^{(0)}_{|n-m|+r} - (n+m+r)\, B^{(0)}_{n+m+r} \} \\ & + \tfrac{1}{2} \sum_{s=1}^{|n-m|} s(|n-m| - s)\, B_s^{(0)}\, B^{(0)}_{|n-m|-s} \\ & - \tfrac{1}{2} \sum_{q=1}^{n+m} q(n+m-q)\, B_q^{(0)}\, B^{(0)}_{n+m-q} \Big] \end{aligned} \qquad \ldots(22)$$

Thus,

$$p = \frac{\eta_0 V L^2}{h^3}\left[\sum_{n=1}^{\infty}\left\{1 - \delta\left(\frac{3\pi^4\bar{\eta}^3}{64}\,\frac{n\pi}{2}\,B_n^{(1)}\right)\right\} B_n^{(0)}\sin\gamma_n x\right] \qquad \ldots(23)$$

where, $\delta = \dfrac{(\lambda_1 - \lambda_2)\,\mu_0 V^2 L^2}{5h^4}$ is a dimensionless parameter similar to

$$N_n = (\lambda_1 - \lambda_2)\,\mu_0 V^2/L^2$$

as introduced by Mow (1968).

The load capacity (W) of the squeeze film is given by

$$W = b \int_0^L p \, dx.$$

Substituting p from eqn. (23) and integrating, we have,

$$W = \frac{\eta_0 b V L^3}{h^3} \left[\sum_{n=1}^{\infty} B_n^{(0)} \left\{ \frac{2}{n\pi} - \delta \left(\frac{3\pi^4 \bar{\eta}^3}{64} B_n^{(1)} \right) \right\} \right] \qquad \ldots(24)$$

or in the dimensionless form, we have,

$$\bar{W} = W(\eta_0 b V L^3 / h^3)^{-1}$$

$$= \sum_{n=1}^{\infty} B_n^{(0)} \left\{ \frac{2}{n\pi} - \delta \left(\frac{3\pi^4 \bar{\eta}^3}{64} B_n^{(1)} \right) \right\}. \qquad \ldots(25)$$

Results and Discussion

The normalized load carrying capacity $\bar{W}$ [see eqn. (25)] has been calculated for various values of different parameters and results are presented in Tables I, II and III. It may be noted that expression (24) with $\delta = 0$ and $\bar{k} = 0$ gives the usual load capacity for a Newtonian fluid with viscosity η between two non-porous plates.

Table I gives the variation of $\bar{W}$ with respect to δ (viscoelastic parameter) and ϕ [concentration parameter, see eqn. (5)] with $\bar{k} = 0$ i.e. for non-porous plates. It has been observed that $\bar{W}$ increases as the concentration parameter ϕ increases, but decreases with the increase in δ. This behaviour has been observed even in the presence of the porous plate.

Tables II and III show the variation of $\bar{W}$ w.r.t. permeability parameter $\bar{k}$ and δ with $\bar{H} = 0.01$ and 0.05 respectively. In these Tables, $\bar{W}$ has been calculated with

TABLE I

Non-dimensional load ($\bar{W}$) for various values of δ and ϕ, with $\bar{k} = 0$ and $\bar{H} = 0$

ϕ \ δ	0	0.001	0.005	0.025
0	1.0000	0.9837	0.9190	0.5952
0.1	1.2500	1.2297	1.1487	0.7440
0.2	1.5000	1.4756	1.3784	0.8927

TABLE II

Non-dimensionless load ($\bar{W}$) *for various values of* δ *and* $\bar{k}$ *with* $\phi = 0.1$ *and* $\bar{H} = 0.01$

$\bar{k}$ \ δ	0	0.001	0.005	0.025
0	1.2500	1.2297	1.1487	0.7440
0.001	1.2497	1.2294	1.1486	0.7441
0.010	1.2480	1.2279	1.1474	0.7451

TABLE III

Non-dimensional load ($\bar{W}$) *for various values of* δ *and* $\bar{k}$ *with* $\phi = 0.1$ *and* $\bar{H} = 0.05$

$\bar{k}$ \ δ	0	0.001	0.005	0.025
0	1.2500	1.2297	1.1487	0.7440
0.001	1.2489	1.2288	1.1481	0.7445
0.005	1.2453	1.2253	1.1456	0.7467

$\phi = 0.1$. It can be seen that $\bar{W}$ decreases as $\bar{k}$ increases for small values of δ (for $\delta = 0, 0.001, 0.005$), however it shows a reversing trend at $\delta = 0.025$ i.e. for $\delta = 0.025$ $\bar{W}$ increases as $\bar{k}$ increases. This trend has also been observed with the increase in thickness (H) of porous facing (compare $\bar{W}$ from Tables II and III with $\bar{k} = 0.001$). It may be remarked here that this observation differs with the conclusions of Tandon and Agarwal, but is in agreement with the results for Newtonian lubricant (Wu 1972, 1978; Sparrow *et al.* (1972)].

REFERENCES

Mow, V. C. (1968). Effects of viscoelastic lubricant on squeeze film lubrication between impinging spheres. *Trans. ASME*, **90**–F, 113.

Tandon, P. N., and Agarwal, Kusum (1979). Behaviour of viscoelastic lubricants in reference to human joints. *Indian J. pure appl. Math.*, **10** (1), 43–50.

Sparrow, E. M., Beavers, G. S., and Hwang, I. T. (1972). Effect of velocity slip on porous walled squeeze films. *Trans. ASME*, **94**–F, 260.

Wu, H. (1972). An analysis of squeeze film between rectangular plates. *Trans. ASME*, **94**–F. 64.

———— (1978). A review of porous squeeze films. *Wear*, **47**, 371.