Indian J. pure appl. Math., **12**(4) : 544–551, April 1981

ANISOTROPIC PROPAGATION OF SONIC WAVES THROUGH THERMALLY CONDUCTING AND RADIATING GASES

SAVITHRI SRINIVASAN

Department of Mathematics, Banaras Hindu University, Varanasi 221005

(*Received* 22 *March* 1980; *after revision* 10 *September* 1980)

The propagation of a sonic discontinuity in an optically thick grey gas at high temperature has been studied by using the ray theory and taking into account the effects of thermal conductivity and thermal radiation. The velocity of the ray and that of propagation of the sonic discontinuity have been determined and interpreted. The fundamental differential equations for the growth of the sonic discontinuity are determined and solved. Explicit solutions for the plane wave are given. It is concluded that if the sonic discontinuity is a compressive wave of order 1, then it terminates into a shock wave after a critical time t_c which has been determined. On the other hand, when the sonic discontinuity is an expansion wave of order 1, then it will decay monotonically and will be damped out ultimately. The results have been computed and graphed in order to show the effects of thermal conduction and thermal radiation on the variations of the strength of the sonic discontinuity during propagation.

1. INTRODUCTION

Using the theory of singular surfaces Thomas (1957a) investigated the propagation of weak discontinuities in ideal gases. He derived the growth equation and discussed the conditions under which the discontinuities either terminate into a shock or decay out. Kaul (1961) studied the propagation of weak discontinuities in ideal gases by considering the entropy and sound speed as dependent variables instead of pressure and density. They both assumed that the medium ahead of the wave is uniform. The study of anisotropic wave propagation has attracted much attention recently and has found many applications (Lighthill 1960, Buchwald 1959, Nigam and Nigam 1962, Moore and Spiegel 1964).

In this investigation we have used the theory of singular surfaces and the method of geometrical optics (ray theory) due to Luneberg (1964). We apply this method to study the anisotropic wave propagation of weak discontinuities through thermally conducting and radiating gases. The medium of the gas is assumed to be optically thick. The fundamental equations governing this system are (Pai 1969):

$$\frac{\partial \rho}{\partial t} + u_i \rho_{,i} + \rho u_{i,i} = 0 \text{ (continuity equation)} \qquad \ldots(1.1)$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j u_{i,j} + (1 + 4R_P)\, p_{,i} - 4 \frac{p}{\rho} R_P \rho_{,i} = 0$$

(momentum equation) ...(1.2)

$$\rho \left(\frac{\partial \eta}{\partial t} + u_i \eta_{,i} \right) T - K_{eff} T_{,ii} - 12 D_R a_R T^2 (T_{,i})^2 = 0$$

(entropy equation) ...(1.3)

$$p = \rho R T \qquad \text{(equation of state)} \qquad ...(1.4)$$

$$T d\eta = \left\{ C_v + 4a_R \frac{T^3}{\rho} \right\} dT - \left\{ \tfrac{4}{3} a_R \frac{T^4}{\rho^2} + \frac{p}{\rho^2} \right\} d\rho \qquad ...(1.5)$$

where the comma followed by an index denotes differentiation with respect to x_i; ρ, p, T, u denote density, pressure, temperature and velocity of the gas respectively; η is the effective entropy in the radiating gas; and R_P is the radiation pressure number given by

$$R_P = \frac{\text{radiation pressure}}{\text{gas pressure}} = \frac{a_R T^4}{3p}$$

and a_R is the Stefan-Boltzmann constant; D_R the Rosseland diffusion coefficient for radiation; K the coefficient of thermal conductivity; and $K_{eff} = K + 4D_R a_R T^3$.

By weak discontinuities we mean that the field variables describing the medium are themselves continuous across it while their derivatives may be discontinuous. Also we have made use of the first and second order compatibility conditions derived by Thomas (1957b) in our investigation.

2. Ray Theory

The ideas presented here originate in the theory of optics. We consider a moving surface $\Sigma(t)$ which is described by the equation of the form: $f(x_1, x_2, x_3, t) = 0$. Let n_i be the components of a unit normal to the surface considered directed into the upstream region so that the normal velocity G of the surface $\Sigma(t)$ will have a positive value. We denote the jump of any quantity z across $\Sigma(t)$ by $[z] = z_1 - z_0$ where z_0 is the value of z ahead of $\Sigma(t)$ and z_1 is the value just behind.

In order to relate the singular surface theory with the ray theory, we first observe that the derivative with respect to time t of any function $F(x_1, x_2, x_3, t)$ along the normal trajectory is given by (Thomas 1957b)

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + F_{,i}\, G n_i \qquad ...(2.1)$$

Since n_i is orthogonal to the wave surface the $\delta/\delta t$ commutes with tangential differentiation. Further if V_i denotes the velocity in the ray direction, then the derivative of F along this direction is given by

$$\frac{d_r F}{dt} = \frac{\partial F}{\partial t} + F_{,i} V_i. \qquad ...(2.2)$$

From (2.1) and (2.2) we obtain

$$\frac{d_r F}{dt} = \frac{\delta F}{\delta t} + F_{,i}(V_i - Gn_i). \qquad ...(2.3)$$

Based on the ray theory (Nariboli 1968) the following relations have been derived

$$V_i = \frac{d_r x_i}{dt} = Gn_i + (\delta_{ij} - n_i n_j) \frac{\partial G}{\partial n_j} \qquad ...(2.4)$$

$$\frac{d_r F}{dt} = \frac{\delta F}{\delta t} + U^{\alpha} F_{,\alpha}. \qquad ...(2.5)$$

3. Propagation of a Sonic Discontinuity

Taking jumps of (1.1) and (1.2) and making use of the first order compatibility conditions, we have

$$(u_n - G)\,\zeta + \rho\lambda = 0 \qquad ...(3.1)$$

$$\rho(u_n - G)\,\lambda_i + (1 + 4R_P)\,\xi n_i - 4\,\frac{p}{\rho}\,R_P \zeta n_i = 0 \qquad ...(3.2)$$

where

$$[u_{i,j}]\, n_j = \lambda_i;\ \ [\rho_{,i}]\, n_i = \zeta;\ \ [p_{,i}]\, n_i = \xi$$

are functions defined over the surface $\Sigma\,(t)$.

From the law of conservation of energy of the fluid for which $K_{eff} \neq 0$ across $\Sigma\,(t)$ we have (Pant and Mishra 1963)

$$[T_{,i}]\, n_i = 0. \qquad ...(3.3)$$

Now differentiating (1.4) with respect to x_i, taking its jump and making use of the compatibility condition of the first order and (3.3) we get

$$\xi = a_1^2\,\zeta \qquad ...(3.4)$$

where a_1 is the isothermal velocity of sound. In consequence of (3.1), (3.2) and (3.4) we get

$$(u_n - G)\,\zeta = -\rho\lambda,\ \ \rho(u_n - G)\,\lambda = \xi \qquad ...(3.5)$$

$$\{(u_n - G)^2 - a_1^2\}\,\zeta = 0. \qquad ...(3.6)$$

Since $\zeta \neq 0$, the relation (3.6) yields

$$G = u_n + a_1. \qquad ...(3.7)$$

Equation (3.7) gives the velocity of propagation of the weak discontinuity.

Differentiating (3.7) with respect to n_j, we get

$$\frac{\partial G}{\partial n_j} = u_j. \qquad ...(3.8)$$

Substituting from (3.8) in (2.4) we get

$$V_i = u_i + a_1 n_i \qquad ...(3.9)$$

which gives the velocity of the ray. The relations (3.7) and (3.9) are the same as those when radiation terms are neglected. Therefore we can conclude that for the case under consideration the radiation effects do not influence the velocity of propagation of a weak discontinuity.

4. Growth Equations Governing the Propagation of a Weak Discontinuity

Differentiating (1.1) and (1.2) with respect to x_k, multiplying by n_k, summing over the index k and taking jumps across $\Sigma(t)$ and using the second order compatibility conditions (Thomas 1957a) we get

$$(u_n - G)\bar{\zeta} + \rho\bar{\lambda} + \frac{\delta\zeta}{\delta t} + U^\alpha \zeta_{,\alpha} + 2\lambda\zeta + \rho g^{\alpha\beta} x_{i,\beta}\lambda_{i,\alpha} = 0 \qquad ...(4.1)$$

$$\rho(u_n - G)\bar{\lambda}_i + (1 + 4R_P)\bar{\xi} n_i + \rho\left(\frac{\delta\lambda_i}{\delta t} + U^\alpha \lambda_{i,\alpha}\right)$$
$$+ (u_n - G)\zeta\lambda_i + g^{\alpha\beta}\xi_{,\alpha} x_{i,\beta}(1 + 4R_P) - 4\frac{p}{\rho}R_P\bar{\zeta} n_i$$
$$- 4\frac{p}{\rho}R_P\zeta_{,\alpha} x_{i,\beta} g^{\alpha\beta} - 12\frac{R_P}{p}\xi^2 n_i + 32\frac{R_P}{\rho}\xi\zeta n_i$$
$$- 20\frac{p}{\rho^2}R_P\zeta^2 n_i = 0 \qquad ...(4.2)$$

where $\bar{\lambda}_i = [u_{i,jk}]\, n_j n_k$, $\bar{\xi} = [p_{,ij}]\, n_i n_j$ and $\bar{\zeta} = [\rho_{,ij}]\, n_i n_j$. Multiplying (4.2) by n_i and summing over i, we get

$$\rho(u_n - G)\bar{\lambda} + (1 + 4R_P)\bar{\xi} + \rho\left(\frac{\delta\lambda_i}{\delta t} + U^\alpha \lambda_{i,\alpha}\right)n_i - 4R_P\frac{p}{\rho}\bar{\zeta}$$
$$+ (u_n - G)\zeta\lambda - 12\frac{R_P}{p}\xi^2 + 32\frac{R_P}{\rho}\xi\zeta - 20R_P\frac{p}{\rho^2}\zeta^2 = 0. \qquad ...(4.3)$$

Equation (1.5) can be written as

$$T\left(\frac{\partial\eta}{\partial t} + u_i\eta_{,i}\right) = \left(C_v + 4a_R\frac{T^3}{\rho}\right)\left(\frac{\partial T}{\partial t} + u_i T_{,i}\right)$$
$$- \left(\tfrac{1}{3}a_R\frac{T^4}{\rho^2} + \frac{p}{\rho^2}\right)\left(\frac{\partial\rho}{\partial t} + u_i\rho_{,i}\right). \qquad ...(4.4)$$

Making use of (1.3) and (4.4) we obtain

$$[T_{,ii}] = \frac{a_1}{K_{eff}} \left\{ \frac{a_R T^4}{3\rho} + a_1^2 \right\} \zeta. \quad \ldots(4.5)$$

Differentiating (1.4) twice with respect to x_i and x_j, multiplying by n_i and n_j taking jump and making use of the compatibility conditions and eqn. (4.5) we get

$$\bar{\xi} = a_1^2 \bar{\zeta} + \frac{\rho R a_1^3 \zeta}{K_{eff}} + \frac{R a_1}{3K_{eff}} a_R T^4 \zeta. \quad \ldots(4.6)$$

Substituting from (4.1) and (4.6) into (4.3) we obtain

$$a_1 \left\{ \frac{\delta\zeta}{\delta t} + U^\alpha \zeta_{,\alpha} + \rho g^{\alpha\beta} \lambda_{i,\alpha} x_{i,\beta} \right\} + \rho \left\{ \frac{\delta \lambda_i}{\delta t} + U^\alpha \lambda_{i,\alpha} \right\} n_i$$

$$+ (1 + 4R_P) \left\{ \frac{\rho R a_1^3 \zeta}{K_{eff}} + \frac{R a_1}{3K_{eff}} a_R T^4 \zeta \right\}$$

$$+ a_1 \zeta \lambda - 12 \frac{R_P}{p} \xi^2 + 32 \frac{R_P}{\rho} \xi \zeta - \frac{20 R_P p}{\rho^2} \zeta^2 = 0. \quad \ldots(4.7)$$

Making use of the relations (2.5) and (3.5) eqn. (4.7) reduces to the form:

$$\frac{d_r \lambda}{dt} + \alpha_0 \lambda + \tfrac{1}{2} \lambda^2 = 0 \quad \ldots(4.8)$$

where

$$\alpha_0 = \tfrac{1}{2} \left\{ (1 + 4R_P) \left[\frac{\rho R a_1^2}{K_{eff}} + \frac{R a_R T^4}{3K_{eff}} \right] - 2a_1 \Omega \right\}.$$

In view of relation (3.5), the growth equations for ζ and ξ can be written as

$$\frac{d_r \zeta}{dt} + \alpha_0 \zeta + \frac{a_1}{2\rho} \zeta^2 = 0 \quad \ldots(4.9)$$

$$\frac{d_r \xi}{dt} + \alpha_0 \xi + \frac{1}{2\rho a_1} \xi^2 = 0. \quad \ldots(4.10)$$

Equations (4.8), (4.9) and (4.10) are the fundamental differential equations for the variations of the quantities λ, ζ and ξ along the normal trajectories of the family of sonic surfaces in the direction of propagation. In view of the relation (3.5) the eqns. (4.9) and (4.10) are derivable from (4.8). Hence the eqn. (4.9) is sufficient to predict the growth or decay of the sonic discontinuity associated with the wave $\Sigma(t)$.

The mean curvature Ω of the sonic wave surface $\Sigma(t)$ is a function of the distance σ and has been calculated as (Lane 1940)

$$\Omega = \frac{\Omega_0 - K_0 \sigma}{1 - 2\Omega_0 \sigma + K_0 \sigma^2}$$

where K_0 and Ω_0 are respectively the Gaussian and mean curvature of the surface $\Sigma(t)$ at time $t = 0$. Substituting for Ω in (4.8) and integrating we get

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} e^{A\sigma/G} \{1 - 2\Omega_0\sigma + K_0\sigma^2\}^{a_1/2G} + \frac{1}{G} e^{A\sigma/G} \{1 - 2\Omega_0\sigma + K_0\sigma^2\}^{a_1/2G} \int e^{A\sigma/G} \{1 - 2\Omega_0\sigma + K_0\sigma^2\}^{-a_1/2G} d\sigma \quad \text{...(4.11)}$$

where

$$A = \tfrac{1}{2}\left\{(1 + 4R_P)\left[\frac{\rho R a_1^2}{K_{eff}} + \frac{R a_R T^4}{3K_{eff}}\right]\right\}$$

λ_0 is the value of λ at the intial time t_0. In order to predict the physical situations more accurately, we consider particular case of a plane wave for which $K_0 = 0$, $\Omega_0 = 0$. In this case, the eqn. (4.9) reduces to the form :

$$\frac{d_r\zeta}{dt} = -A\zeta - \frac{a_1}{2\rho}\zeta^2 \quad \text{...(4.12)}$$

the solution of which can be written as

$$\zeta = \frac{\zeta_0}{e^{At} + a_1\zeta_0(e^{At} - 1)(2\rho A)^{-1}} \quad \text{...(4.13)}$$

where ζ_0 is the initial discontinuity.

The solution (4.13) suggests that if the initial sonic discontinuity is an expansion wave of order 1, ($\zeta_0 > 0$), then the discontinuity will decay monotonically and will be damped out ultimately.

Rewriting the eqn. (4.14) in the non-dimensional form, we have

$$\delta = \frac{1}{e^{\eta} + A_0(e^{\eta} - 1)} \quad \text{...(4.14)}$$

where

$$\rho = \frac{\zeta}{\zeta_0},\ \eta = \frac{A\sigma}{G},\ A_0 = \frac{a_1\zeta_0}{2\rho A}$$

The nature of variations of the strength of the sonic discontinuity for typical cases is shown in Fig. 1.

It is now obvious from Fig. 1 that the effects of thermal radiation are to cause more rapid damping effects on the sonic wave propagation.

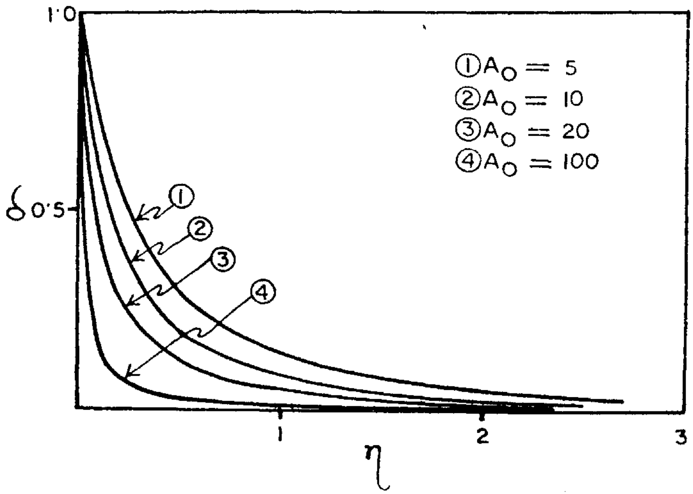
1·0
δ 0·5
① A_0 = 5
② A_0 = 10
③ A_0 = 20
④ A_0 = 100
1
2
3
η

FIG. 1.

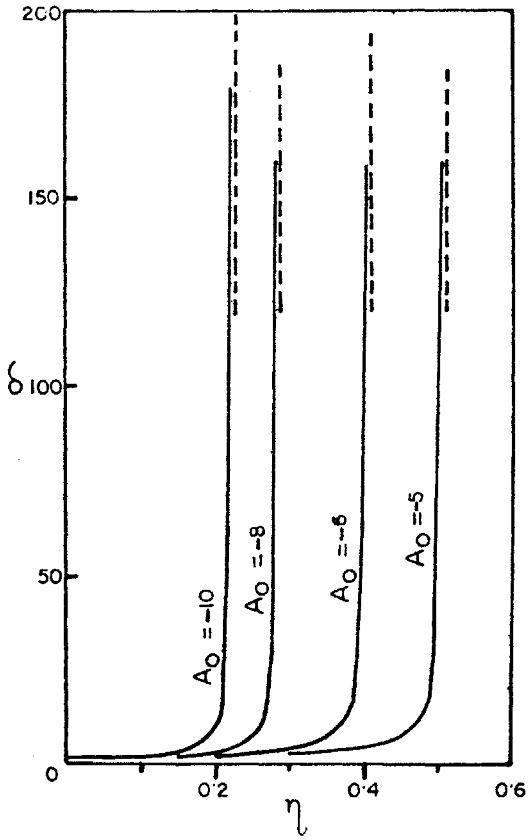
200
150
δ 100
50
0
0·2
0·4
0·6
η
A_0 = −10
A_0 = −8
A_0 = −6
A_0 = −5

FIG. 2.

On the other hand, if the initial sonic discontinuity is a compressive wave of order 1, ($\zeta_0 < 0$), the continuity of density across a wave will break down. Hence in this case a sonic discontinuity will grow and after a critical time t_c given by

$$t_c = t_0 + \frac{1}{A} \log \frac{a_1 \mid \zeta_0 \mid}{2\rho A + a_1 \mid \zeta_0 \mid}$$

it will terminate into a shock wave.

For typical cases the effects of thermal conduction and radiation on the growth of the sonic discontinuity and its termination into a shock wave have been shown in Fig. 2.

It is interesting to note that the critical time t_c decreases for the increasing values of A_0. Hence the effects of thermal conduction and radiation on the propagation of a compressive wave are to decrease the duration of time by which a sonic discontinuity will terminate into a shock wave.

From the similarity of eqns. (4.8), (4.9) and (4.10) it is obvious that the other discontinuities λ and ξ will also behave in a similar manner.

References

Buchwald, V. T. (1959). Elastic waves in anisotropic media. *Proc. R. Soc.*, A **253**, 563–80.

Kaul, C. N. (1961). On singular surfaces of order one in ideal gases. *J. Math. Mech.*, **10**.

Lane, S. P. (1940). Metric Differential Geometry of Curves and Surfaces. University of Chicago Press, Chicago, p. 209.

Lighthill, M. J. (1960). Studies in magnetohydrodynamic waves and other anisotropic wave motions. *Phil. Trans. R. Soc.*, **252** A, 397–430.

Luneberg, R. K. (1964). Mathematical Theory of Optics. University of California Press, Berkeley.

Moore, D. K., and Spiegel, E. A. (1964). Waves in a compressible atmosphere. *Astr. J.*, **139**, 48–71.

Nariboli, G. A. (1968). On some aspects of wave propagation. *J. Math. Phys. Sci.*, **3**, 294–310.

Nigam, S. D., and Nigam, P. D. (1962). Wave propagation in rotating liquids. *Proc. R. Soc.*, A **266**, 247–56.

Pai, S. I. (1969). Inviscid flow of radiation gas dynamics. *J. Math. Phys. Sci.* **3**, 361–70.

Pant, J. C., and Mishra, R. S. (1963). Shock waves of finite thickness in magneto gas dynamics. *Rendiconti Cer. Math. Dipalermo*, Ser. II, Tomo XII, 1–23.

Thomas, T. Y. (1957a). The growth and decay of sonic discontinuities in ideal gases. *J. Math. Mech.*, **9**, 455–70.

——— (1957b). Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics. *J. Math. Mech.*, **6**, 311–22.