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CARTESIAN PRODUCT OF TWO MANIFOLDS

H. B. PANDEY

Mathematics Department, R.B.S. College, Agra

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Through the present note author wants to study the Cartesian product of two manifolds. In section 1, the C^∞ forms, connections and other preliminary definition, have been given. Section 2 deals with the properties of $M_1 \times M_2$. Section 3 is equipped with the curvature tensor of $M_1 \times M_2$. It has been shown that the curvature tensor of type (0, 4) of $M_1 \times M_2$ is the sum of the curvature tensors of type (0, 4) of M_1 and M_2. In the last a simple example has been taken.

1. INTRODUCTION

Let M_1 and M_2 be two C^∞ manifolds of dimensions n_1 and n_2 respectively. Let $(M_1)_{m_1}$ and $(M_2)_{m_2}$ be the tangent spaces of M_1 and M_2 at $m_1 \in M_1$ and $m_2 \in M_2$ respectively. Let these be the vector spaces over field C (complex field). It is known that (Hicks 1965)

$$(M_1 \times M_2)\,(m_1, m_2) \cong (M_1)_{m_1} \times (M_2)_{m_2}.$$

Let us define

$$(X_1, X_2) + (Y_1, Y_2) = (X_1 + Y_1, X_2 + Y_2) \qquad \text{...(1.1)}$$

and

$$\lambda(X_1, X_2) = (\lambda X_1, \lambda X_2) \qquad \text{...(1.2)}$$

where X_1, Y_1 etc. belong to $(M_1)_{m_1}$ and X_2, Y_2 belonging $(M_2)_{m_2}$, λ is scalar, say in C. Under the operations defined in (1.1) and (1.2) we can show that $(M_1)_{m_1} \times (M_2)_{m_2}$ is a vector space over field C.

Now let us define

$$F : (M_1)_{m_1} \times (M_2)_{m_2} \rightarrow (M_1)_{m_1} \times (M_2)_{m_2}$$

as

$$F(X_1, X_2) = (F_1X_1, F_2X_2) \qquad \text{...(1.3)}$$

where F_1 and F_2 are linear transformations on $(M_1)_{m_1}$ and $(M_2)_{m_2}$ respectively. It is easy to verify that F is also a linear transformation over $(M_1)_{m_1} \times (M_2)_{m_2}$. The dimension of $M_1 \times M_2$ is the sum of dimensions of M_1 and M_2.

Let f_1, f_2 be C^∞ functions over M_1 and M_2 respectively. Let us define (f_1, f_2) a C^∞ function over $M_1 \times M_2$. Thus (f_1, f_2) is C^∞ at each point of $M_1 \times M_2$.

Let us define

$$(X_1, X_2)(f_1, f_2) \overset{def}{=} (X_1 f_1, X_2 f_2). \qquad \ldots(1.4)$$

Let D_1 and D_2 be connexions on M_1 and M_2. Let D be a connexion on $(M_1)_{m_1} \times (M_2)_{m_2}$ defined as

$$D_{(X_1, X_2)}(Y_1, Y_2) = (D_{1X_1} Y_1, D_{2X_2} Y_2). \qquad \ldots(1.5)$$

Since D defined by (1.5) satisfies the following four properties, hence D is an infinitesimal connexion :

(i) $D_{(X_1,X_2)}\{(Y_1 Y_2) + (Z_1, Z_2)\} = D_{(X_1,X_2)}(Y_1, Y_2) + D_{(X_1,X_2)}(Z_1, Z_2)$

(ii) $D_{\{(X_1,X_2)+(Y_1,Y_2)\}}(Z_1, Z_2) = D_{(X_1,X_2)}(Z_1, Z_2) + D_{(Y_1,Y_2)}(Z_1, Z_2)$

(iii)
$$\begin{aligned} D_{\{(f_1,f_2)(X_1,X_2)\}}(Y_1, Y_2) &= D_{(f_1X_1, f_2X_2)}(Y_1, Y_2) \\ &= (f_1 D_{1X_1} Y_1, f_2 D_{2X_2} Y_2) \\ &= (f_1, f_2)(D_{1X_1} Y_1, D_{2X_2}, Y_2) \\ &= (f_1, f_2) D_{(X_1,X_2)}(Y_1, Y_2) \end{aligned}$$

(iv)
$$\begin{aligned} D_{(X_1,X_2)}(f_1, f_2)(Y_1, Y_2) &= D_{(X_1,X_2)}(f_1 Y_1, f_2 Y_2) \\ &= (D_{1X_1} f_1 Y_1, D_{2X_2} f_2 Y_2) \\ &= (X_1 f_1) Y_1 + f_1 D_{1X_1} Y_1 (X_2 f_2) Y_2 + f_2 D_{2X_2} Y_2) \\ &= ((X_1, X_2)(f_1, f_2))(Y_1, Y_2) \\ &\quad + (f_1, f_2)(D_{1X_1} Y_1, D_{2X_2} Y_2). \end{aligned}$$

2. Properties

Theorem 2.1 — $M_1 \times M_2$ is almost complex if and only if M_1 and M_2 are almost complex manifolds.

PROOF : Let M_1 and M_2 be almost complex manifolds. Then from (1.3), we have

$$\begin{aligned} F^2(X_1, X_2) &= (F_1^2 X_1, F_2^2 X_2) \\ &= -(X_1, X_2) \end{aligned}$$

hence $M_1 \times M_2$ is almost complex. Converse is easy to check.

Let us define a Riemannian metric 'g' over $M_1 \times M_2$ as :

$$g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + g_2(X_2, Y_2). \quad ...(2.1)$$

where g_1 and g_2 are Riemannian metrics over M_1 and M_2 respectively. The definition (2.1) is justified, for both the sides of (2.1) are scalars in C. Moreover as g_1 and g_2 are Riemannian metrics so is 'g', (g is symmetric, linear and positive definite). If we replace '$+$' by '$.$' (sign of multiplication of C), then g will not be linear and hence will not be Riemannian. Hence a Riemannian metric in $M_1 \times M_2$ may be defined as (2.1)

It may be easily verified that g is Hermitian if and only if g_1 and g_2 are Hermitian.

Theorem 2.2 — $M_1 \times M_2$ is Kählerian if and only if M_1 and M_2 are Kählerian.

PROOF : Let M_1 and M_2 be Kählerian, then

$$D_{1X_1} F_1 = 0 \quad \text{and} \quad D_{2X_2} F_2 = 0. \quad ...(2.2)$$

Let
$$D_{(X_1,X_2)} F(Y_1, Y_2) = D_{(X_1,X_2)}(F_1Y_1, F_2Y_2)$$
$$= F(D_{1X_1} Y_1, D_{2X_2} Y_2) = FD_{(X_1,X_2)}(Y_1, Y_2)$$

which shows that

$$(D_{(X_1}, F_{X_2)}) (Y_1, Y_2) = 0 \quad ...(2.3)$$

hence $M_1 \times M_2$ is Kählerian. Conversely let (2.3) be given then simple calculations provide

$$0 = ((D_{1X_1} F_1) Y_1, (D_{2X_2} F_2) Y_2)$$

which completes the proof.

Theorem 2.3 — $M_1 \times M_2$ is almost Tachibane if and only if M_1 and M_2 are almost Tachnibana.

PROOF : Let M_1 and M_2 be almost Tachibana. Let us construct

$$(D_{(X_1,X_2)}F)(Y_1, Y_2) + FD_{(X_1,X_2)}(Y_1, Y_2)$$
$$+ (D_{(Y_1,Y_2)} F)(X_1, X_2) + FD_{(Y_1,Y_2)}(X_1, X_2)$$
$$= ((D_{1X_1} F_1) Y_1 + (D_{1Y_1} F_1) X_1 + F_1D_{1X_1} Y_1$$
$$+ F_1D_{1Y_1} X_1, (D_{2X_2} F_2) Y_2 + (D_{2Y_2} F_2)(X_2)$$
$$+ F_2D_{2X_2} Y_2 + F_2D_{2Y_2} X_2)$$

(equation continued on p. 58)

$$= FD_{(X_1,X_2)}(Y_1, Y_2) + FD_{(Y_1,Y_2)}(X_1, X_2) \quad \text{...(2.4)}$$

which shows that $M_1 \times M_2$ is almost Tachibana.

Conversely let $M_1 \times M_2$ be almost Tachibana. Then left hand side of (2.4) is

$$FD_{(X_1,X_2)}(Y_1, Y_2) + FD_{(Y_1,Y_2)}(X_1, X_2)$$
$$= ((D_{1X_1} F_1)\, Y_1 + (D_{1Y_1} F_1)\, X_1, (D_{2X_2} F_2)\, Y_2$$
$$+ (D_{2Y_2} F_2)\, X_2) + FD_{(X_1,X_2)}(Y_1, Y_2) + FD_{(Y_1,Y_2)}(X_1, X_2)$$

which shows that

$$(D_{iX_i} F_i)\, Y_i + (D_{iY_i} F_i)\, X_i = 0,$$

for $i = 1, 2$ thus M_1 and M_2 are almost Tachibana.

If M_1 and M_2 are almost contact manifold, then $M_1 \times M_2$ may not be so, as the dimension of $M_1 \times M_2$ will not be necessarily odd.

Similar properties hold for GF-structures, f-structures and $\phi(4, \pm 2)$-structures also.

3. Curvature

It is easy to verify that

$$[(X_1, X_2), (Y_1, Y_2)]\,(f_1, f_2) = (X_1, X_2)\,((Y_1, Y_2)\,(f_1, f_2))$$
$$- (Y_1, Y_2)\,((X_1, X_2)\,(f_1, f_2))$$
$$= ([X_1, Y_1] f_1, (X_2, X_2] f_2)$$
$$= ([X_1, Y_1], [X_2, Y_2])\,(f_1, f_2).$$

Thus

$$[(X_1, X_2, (Y_1, Y_2)] = ([X_1 Y_1], [X_2, Y_2]).$$

Let $K_i(X_i, Y_i, Z_i)$, $i = 1, 2$ be two curvature tensors in M_i, $i = 1, 2$ respectively. Let $K(X, Y, Z)$ be the curvature tensor in $M_1 \times M_2$. In the definition of $K(X, Y, Z)$ let us write (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) for X, Y, Z respectively. Applying the definition for connexion we get

$$K(X, Y, Z) = (K_1(X_1, Y_1, Z_1), K_2(X_2, Y_2, Z_2)) \quad \text{...(3.1)}$$

$$'K(X, Y, Z, W) \overset{def}{=} g(K(X, Y, Z), W)$$
$$= {'K_1}(X_1, Y_1, Z_1, W_1) + {'K_2}(X_2, Y_2, Z_2, W_2) \quad \text{...(3.2)}$$

Thus the curvature tensors $M_1 \times M_2$ are given by (3.1) and hence by (3.2) (Ficken 1939).

It can be easily verified that if M_1 is of constant curvature then is $M_1 \times M_2$ also.

In light of (3.2) it is easy to verify that Ricci tensor of $M_1 \times M_2$ is sum of Ricci tensors of M_1 and M_2.

Let $M_1 \times M_2$ be Einstein space. Then

$$\text{Ric}(X, Y) = C \quad g(X, Y), \qquad \ldots(3.3)$$

where $C = \dfrac{K}{n}$, K is the scalar curvature of $M_1 \times M_2$.

From (3.3) it is easy to verify that

$$\text{Ric}_i(X_i, Y_i) = C \quad g_i(X_i, Y_i).$$

$i = 1, 2$. Hence $C = \dfrac{K}{n} = \dfrac{K_1}{n_1} = \dfrac{K_2}{n_2}$.

Conversely let M_1 and M_2 be Einstein and $\dfrac{K}{n} = \dfrac{K_1}{n_1} = \dfrac{K_2}{n_2}$, then easy to verify that $M_1 \times M_2$ is Einstein. Thus we have, $M_1 \times M_2$ is an Einstein space if and only if M_1 and M_2 are also and the scalar curvature of $M_1 \times M_2$, M_1, M_2 are in ratio $n : n_1 : n_2$ where n, n_1, n_2 are the respective dimensions.

It can be easily checked that $K = K_1 + K_2$ provided $M_1 \times M_2$ is Einstein.

Example — Let us take two mutually perpendicular straight lines showing R. Then the space formed by them is 2-dimensional $R \times R$, a Cartesian plane.

If one defines $f : R \to R$ as $f(x) = -x$, then $f^2(x) = x$, hence f may be treated as almost product structure over R. Let

$$F = R \times R \to R \times R$$

$$F(x, y) = (f(x), f(y)).$$

We can check that F is almost product structure on $R \times R$.

For metric let 1 and 3 be the position vectors on $R \rightarrow$ and take 2, 4 the p.v. on $R \uparrow$. Then $p(1, 2)$ and Q (3, 4) are the position vectors in $R \times R$.

$$g((1, 2), (3, 4) = 5\sqrt{5} \cos\theta. \qquad \ldots(3.6)$$

It may be calculated that $\theta = 10.3°$. Hence

$$g((1, 2), (3, 4)) = 10.9. \qquad \ldots(3.7)$$

Again consider

$$g_1(1, 3) = 3 \cos\theta = 3, \qquad \ldots(3.8)$$

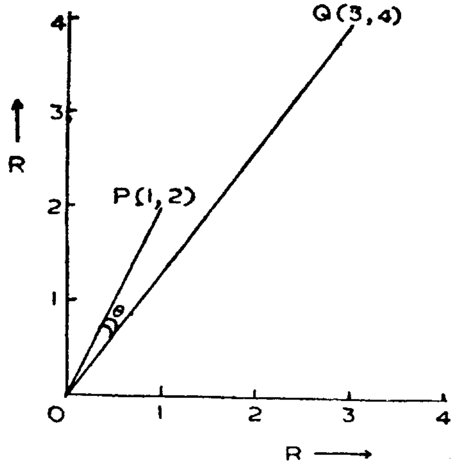

FIG. 1.

$$g_2(2, 4) = 8 \cos \theta = 8, \qquad ...(3.9)$$

hence

$$g((1, 2), (3, 4)) = g_1(1, 3) + g_2(2, 4) \qquad ...(3.10)$$

the minor difference between the two sides of (3.10) is due to round off error. Thus (F, g) constructs an almost product structure on $R \times R$.

If one multiplies g_1 and g_2, then product is not equal to g.

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