Indian J. pure appl. Math., 12(4) : 552–557, April 1981

ELASTIC-PLASTIC TRANSITIONS IN TRANSVERSELY ISOTROPIC SHELLS UNDER UNIFORM PRESSURE

SURESH HULSURKAR
Department of Mathematics, Indian Institute of Technology, Kharagpur 721302

(*Received* 7 *May* 1980)

Seth's transition theory of elastic-plastic and creep deformations is exploited to study the elastic-plastic transitions in transversely isotropic shells under uniform pressure. The transition theory approach does not require the assumptions of yield criteria and the associated flow rules. The expressions for stresses in a transition state are derived which lead to those valid for the fully plastic state. These results are in close agreement with the classical results.

1. INTRODUCTION

The transition approach to deformations in solids has been developed by Seth (1963, 1964, 1966, 1974) and has been successfully applied by various researchers to a large number of problems in elastic-plastic and creep transitions. In Seth's theory the major difference from the classical theories of plasticity and creep arises from the rejection of the ad-hoc assumptions about the yield conditions and flow rules. However, it can be shown that the results obtained by transition theory do satisfy the so-called yield condition in plastic deformations. In contrast to the work on elastic-plastic and creep transitions in isotropic bodies very little has been done on anisotropic bodies. In this paper we solve the problem of elastic-plastic transitions in transversely isotropic shells under uniform internal pressure. The ideas employed in this paper all stem from Seth's paper (1970) on transition phenomena in anisotropic bodies. The results obtained here are in close agreement with those coming through classical methods.

2. STRESS-STRAIN RELATIONS

The transition approach for anisotropic bodies has been developed thoroughly by Seth (1970). In this paper we shall deal with transversely isotropic materials. Hence we summarize here only those results which we need later and emphasize that these are only the special cases of the general results established by Seth (1970). We fix an orthogonal coordinate system with respect to which any point is represented by (x_1, x_2, x_3). We assume that the coordinate system chosen is such that the x_3-axis coincides with the axis of the elastic symmetry in the material. In other words the transverse isotropy is assumed to be about the x_3-axis. In this case the stress-strain relations are (Lekhnitiskii 1963).

$$\left.\begin{aligned} T_{11} &= c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} \\ T_{22} &= c_{12}e_{11} + c_{11}e_{22} + c_{13}e_{33} \\ T_{33} &= c_{13}e_{11} + c_{13}e_{22} + c_{33}e_{33} \\ T_{23} &= c_{44}e_{23},\ T_{13} = c_{44}e_{13},\ T_{12} = c_{66}e_{12} \end{aligned}\right\} \quad \ldots(2.1)$$

where T_{ij}, e_{ij} and c_{ij} denote respectively the stress tensor, strain tensor and the elastic response coefficients with $2c_{66} = c_{11} - c_{12}$.

A solid body subjected to external forces first deforms elastically and if the loads are increased sufficiently then a transition state is reached which may lead to a fully plastic state. When these transitions are taking place the values of c_{ij} will be changing continuously and a fully plastic state will be characterized by a condition on them. Seth (1970) has shown that a fully plastic state is given by $\nu_{ij} + \nu_{ji} = 1$, $i \neq j$ where ν_{ij}'s are generalized Poisson's ratios. This also lead to expressions for yield stresses. These results can be stated in a simpler form if we use the following notations suggested by the isotropic case:

$$\left.\begin{aligned} &\alpha_1 = (c_{11} - c_{12})/c_{11},\ \alpha_2 = (c_{33} - c_{13})/c_{33}, \\ &c_{11} + c_{12} - 2c_{13} = c_{33}\,\alpha_3\,(1 - \alpha_2). \end{aligned}\right\} \quad \ldots(2.2)$$

Then the required condition is equivalent to α_1, α_2, α_3 independently tending to zero. With these notations the yield stress Y_3 in x_3-direction is given by

$$Y_3 = \frac{\mu_3(3 - \alpha_2 - 2\alpha_3)}{(2 - \alpha_3 - \alpha_2\alpha_3)} \quad \ldots(2.3)$$

where $\mu_3 = (c_{33} - c_{13})$. For isotropic bodies $\alpha_2 = 0$, $\mu_3 = \mu$, $\alpha_3 = 2\mu/(\lambda + 2\mu) = c$, where λ, μ are Lame's constants, and we get the expressions given by Seth (1974). We can similarly find the expressions for yield stresses Y_1, Y_2. For further details we refer to Seth (1970).

3. Spherical Shell

We consider an anisotropic spherical shell of constant thickness under uniform internal pressure. The anisotropy is assumed to be of spherical type which means there is a transverse isotropy about the radius vector. The symmetry in the elastic properties and the loading permits us to take the displacements in spherical coordinates (r, θ, ϕ) as

$$u = r(1 - P),\ v = 0,\ w = 0 \quad \ldots(3.1)$$

where P is a function of r only.

The components of the finite strain tensor are given by (Seth 1963)

$$\left. \begin{aligned} e_{rr} &= \tfrac{1}{2}\{1 - (rP' + P)^2\} \\ e_{\theta\theta} &= e_{\phi\phi} = \tfrac{1}{2}(1 - P^2) \\ e_{r\theta} &= e_{\theta\phi} = e_{r\phi} = 0 \end{aligned} \right\} \qquad \ldots(3.2)$$

where prime denotes differentiation with respect to r.

The components of the stress tensor, after identifying x_3-direction with that of r in equation (2.1), are given by

$$\left. \begin{aligned} T_{rr} &= c_{33}e_{rr} + 2c_{13}e_{\theta\theta} \\ T_{\theta\theta} &= T_{\phi\phi} = c_{13}e_{rr} + 2(c_{11} - c_{66})\, e_{\theta\theta} \\ T_{r\theta} &= T_{\theta\phi} = T_{r\phi} = 0 \end{aligned} \right\} \qquad \ldots(3.3)$$

where the strains e_{rr}, $e_{\theta\theta}$ are given by eqn. (3.2).

The equation of equilibrium

$$\frac{\partial T_{rr}}{\partial r} + \frac{2(T_{rr} - T_{\theta\theta})}{r} = 0 \qquad \ldots(3.4)$$

leads to the following non-linear differential equation to be satisfied by P

$$(P + rP')P'' + \left(3 - \frac{c_{13}}{c_{33}}\right) P'^2 + \frac{4PP'}{r} - \frac{1 - P^2}{r^2} \times \left(1 + \frac{c_{13}}{c_{33}} - \frac{2c_{11} - 2c_{66}}{c_{33}}\right) = 0. \qquad \ldots(3.5)$$

Now we use the substitutions given by eqn. (2.2) and put $rP' = PU$ in eqn. (3.5) to get

$$\{U^3 + (2 + \alpha_3)\, U^2 + 3U - (1 - P^2)\,(\alpha_2\alpha_3/P^2)\} \frac{dP}{dU} + PU(U + 1) = 0. \qquad \ldots(3.6)$$

This shows that the transition points of P are

$$U = -1, \quad U = \pm\,\infty.$$

Now for elastic-plastic transitions we should consider the transition through T_{rr} at the transition point $U = \pm\infty$. With this in view we define R by

$$R = 3 - 2\alpha_3 - \frac{2T_{rr}}{c_{33}} \qquad \ldots(3.7)$$

From eqn. (3.2) and (3.3) we get

$$R = P^2\,\{(U + 1)^2 + 2(1 - \alpha_3)\} \qquad \ldots(3.8)$$

Differentiating eqn. (3.8) and using eqn. (3.6), we get

$$\frac{d \log R}{d \log r} = \frac{-2\alpha_3 U(U+2) + (2\alpha_2\alpha_3/P^2)(1-P^2)}{(U+1)^2 + 2(1-\alpha_3)}$$

which gives

$$\frac{d \log R}{d \log r} = -2\alpha_3 \text{ as } U \to \pm\infty. \qquad \ldots(3.9)$$

From eqns. (3.7) and (3.9), we have

$$T_{rr} = \frac{c_{33}}{2}(3 - 2\alpha_3 - Ar^{-2\alpha_3}) \qquad \ldots(3.10)$$

where A is a constant.

The boundary conditions for a spherical shell under internal pressure p are

$$T_{rr} = -p \text{ at } r = a,$$

$$T_{rr} = 0 \text{ at } r = b, \qquad \ldots(3.11)$$

where a and b are respectively the internal and external radii of the shell. From equations (2.3), (3.4), (3.10) and (3.11) we get the following expressions for a transition state:

$$\left.\begin{aligned} T_{rr} &= \frac{Y_3(2-\alpha_3-\alpha_2\alpha_3)(3-2\alpha_3)}{(3-\alpha_2-2\alpha_3)\alpha_3}\left\{1-\left(\frac{b}{r}\right)^{2\alpha_3}\right\} \\ T_{\theta\theta} - T_{rr} &= \frac{Y_3(2-\alpha_3-\alpha_2\alpha_3)(3-2\alpha_3)}{(3-\alpha_2-2\alpha_3)}\left(\frac{b}{r}\right)^{2\alpha_3} \\ P_o &= \frac{Y_3(2-\alpha_3-\alpha_2\alpha_3)(3-2\alpha_3)}{(3-\alpha_2-2\alpha_3)\alpha_3}\left\{\left(\frac{b}{a}\right)^{2\alpha_3}-1\right\}. \end{aligned}\right\} \qquad \ldots(3.12)$$

Here p_o denotes the pressure at which a transition from an initially elastic state to a plastic one begins. Now for a fully plastic state $\alpha_2, \alpha_3 \to o$ and this reduces the expressions in eqn. (3.12) to the ones given by Seth (1963) for isotropic materials. This is to be expected because in the classical approach the same results are obtained even with the most general form of the yield condition proposed by Hill (1950).

4. Cylindrical Shell

Now we consider a hollow cylindrical shell under uniform internal pressure. Here we investigate two cases. In one case we assume that there is transverse isotropy about the axis of the cylinder and in the other about the radius vector. In either case the symmetry in the problem allows us to take the displacements in cylindrical coordinates $(r, \theta\ z)$ as

$$u = r(1-Q),\ v = 0,\ w = 0 \qquad \ldots(4.1)$$

where $Q = Q(r)$.

The finite strains are given by

$$\left.\begin{aligned} e_{rr} &= \tfrac{1}{2}\{1 - (rQ' + Q)^2\} \\ e_{\theta\theta} &= \tfrac{1}{2}(1 - Q^2) \\ e_{zz} &= e_{r\theta} = e_{\theta z} = e_{rz} = 0 \end{aligned}\right\} \qquad \ldots(4.2)$$

where $Q' = \dfrac{dQ}{dr}$.

The equation of equilibrium to be satisfied is

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0. \qquad \ldots(4.3)$$

Case 1 — Here we assume the transverse isotropy to be about the axis of the cylinder. Hence identifying the x_3-direction with that of z-axis, eqn. (2.1) gives

$$\left.\begin{aligned} T_{rr} &= c_{11}e_{rr} + (c_{11} - 2c_{66})\, e_{\theta\theta} \\ T_{\theta\theta} &= (c_{11} - 2c_{66})\, e_{rr} + c_{11}e_{\theta\theta} \\ T_{zz} &= c_{13}e_{rr} + c_{13}e_{\theta\theta} \\ T_{r\theta} &= T_{\theta z} = T_{rz} = 0 \end{aligned}\right\} \qquad \ldots(4.4)$$

where e_{rr}, $e_{\theta\theta}$ are given by eqn. (4.2).

From eqn. (4.3) and (4.4), we get

$$(Q + rQ')\, Q'' + \left(2 + \frac{c_{66}}{c_{11}}\right) Q'^2 + \frac{3QQ'}{r} = 0. \qquad \ldots(4.5)$$

Now the substitution $rQ' = QV$ and eqn. (2.2) reduces eqn. (4.5) to

$$\left\{V^2 + \left(2 + \frac{\alpha_1}{2}\right) V + 2\right\} \frac{dQ}{dV} + Q(V + 1) = 0 \qquad \ldots(4.6)$$

which shows that the transition points of Q are

$$V = -1,\ V = \pm\,\infty.$$

Again we can define R_1 similar to R given by eqn. (3.7) to get the transitions through T_{rr} for the transition point $V = \pm\,\infty$. Thus the transitional stresses can be obtained by the method similar to that described in the earlier section. We note that the eqn. (4.6) governing the transitions is similar to that of isotropic cylinder. Hence we can expect that this will lead to the solutions which are similar to the ones obtained for isotropic materials. This is what exactly happens and we leave the details to the reader. For a fully plastic state this has also been shown by Hu (1956) with the help of Hill's yield condition.

Case 2 — In this case assuming the transverse isotropy about the radius vector, we get

$$\left.\begin{aligned} T_{rr} &= c_{33}e_{rr} + c_{13}e_{\theta\theta} \\ T_{\theta\theta} &= c_{13}e_{rr} + c_{11}e_{\theta\theta} \\ T_{zz} &= c_{13}e_{rr} + (c_{11} - 2c_{66})\, e_{\theta\theta} \\ T_{r\theta} &= T_{\theta z} = T_{rz} = 0 \end{aligned}\right\} \qquad \ldots(4.7)$$

where e_{rr}, $e_{\theta\theta}$ are given by eqn. (4.2). Substituting this in eqn. (4.3) and using eqn. (2.2) and the substitution $rQ' = QV$, we get

$$\left\{V^3 + \left(5 - \frac{\alpha_3}{2}\right) V^2 + 4V - (1 - Q^2)\,(\alpha/2Q^2)\right\} \frac{dQ}{dV} + QV(V+1) = 0 \qquad \ldots(4.8)$$

where $\alpha = (2 - \alpha_1 - 3\alpha_3 + \alpha_1\alpha_3 + \alpha_2\alpha_3)/(2 - \alpha_1)$.

This is the differential equation governing the transitions with the transition points $V = -1$, $V = \pm\infty$. This equation is similar to eqn. (3.6), and leads to the results which are of the same form as that of an isotropic cylinder. The transitional stresses can be easily derived by the method described earlier and therefore we omit the details.

References

Hill, R. (1950). The Mathematical Theory of Plasticity. Clarendon Press, Oxford.

Hu, L. (1956). Studies on plastic flow of anisotropic metals. *J. appl. Mech.*, **23**, 446.

Lekhnitiskii, S. G. (1963). Theory of Elasticity of an Anisotropic Elastic Body. Holden Day, San Francisco.

Seth, B. R. (1963). Elastic-plastic transition in shells and tubes under pressure. *ZAMM*, **43**, 345.

——— (1964). Transition Theory of Elastic-plastic Deformation. Bangalore Press, Bangalore 1964.

——— (1966). Generalized strain and transition concepts for elastic-plastic deformation, creep and relaxation. *Proc. XI Inter. Cong. Appl. Mech., Munich*, p. 383.

——— (1970). Transition problems of Aelotropic yield and creep rupture. *Inter. Centre for Mech. Sci., Courses and Lectures No.* 47, *Udine*, Springer-Verlag, New York.

——— (1974). Creep-plastic effects in sheet bending. *ZAMM*, **54**, 557.