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INVENTORY MODEL WITH UNKNOWN DEMAND DISTRIBUTION*

S. P. AGGARWAL AND VIJAY P. GOEL**

Department of Operational Research, University of Delhi, Delhi 110007

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In this paper a single-item inventory model is discussed when T, the inter-arrival time of demands, follows some general (known) probability distribution and the quantity in each demand follows an unknown probability distribution with partial information. After every N periods, each of length T, an order is placed for an amount which brings the sum of stocks on hand and on order up to a fixed level Q. In this article, almost optimal values for N and Q are obtained.

1. INTRODUCTION

The knowledge of the demand distribution is based on a number of factors, namely the sequence of past demands, judgement about trends, irregularities and seasonality, etc. For various reasons these factors may be insufficient to estimate the future distribution. Most of the recent work in mathematical inventory theory has assumed that the demand-distribution is completely known. In this paper a single-warehouse single-stage single-item inventory model is studied in which the inter-demand time and quantity demanded are independent random variables with known inter-demand time distribution and unknown demand distribution. It is further assumed that only partial information, viz. the mean and the variance of the (unknown) demand distribution, is given. These may be either estimated from past history or determined in other ways. Consumption during a demand-interval may cause depletion of the inventory. The cumulative demands in N successive periods form a sequence of independent identically distributed random variables.

In this article an inventory model is discussed in which the stock of the system is reviewed after every N periods. At the review time an order is placed for an amount which brings the sum of stock on hand and on order up to the level Q. Almost optimal values of N and Q are obtained so as to maximize the long term expected revenue per inter-demand period. These problems occur frequently in defence budgeting in respect of stocking of spare parts and perishable goods.

Here a model of the order cycle system is formulated and revenue function is developed. The approach is to minimize this revenue function with respect to all

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**Present address: Department of Statistics, Khalsa College (Bombay Univ.), Bombay 400019.

distributions which possess the given mean and variance. The minimised revenue function is then maximized with respect to N and Q.

2. Preliminaries and Notations

T = Length of one period (known random variable)

Q = Stock on hand plus on order (unknown)

N = Number of periods (unknown)

C = Cost of an item, independent of the number purchased

W = Warehouse cost of maintaining the storage for one unit of stock during a period T

A = Fixed cost of order and review for N periods

r = Selling price of an item; it is independent of the number sold.

3. Formulation of the Mathematical Model

Let $T_1, T_2, \ldots, T_n$, the inter-arrival times of the 1st, 2nd, ..., nth demands, be independently and identically distributed with probability density $f(t)$ and further let the quantities $x_1, x_2, \ldots, x_n$ at the respective demands be independently and identically distributed random variables with unknown probability density $l(x)$.

Let η_t denote the number of demands occurring in time t, then

$$\eta_t < R \text{ iff } S_R > t \text{ where } S_R = \sum_{i=1}^{R} T_i$$

so that $$Pr(\eta_t < R) = Pr(S_R > t) = 1 - F_R(t)$$

where $F_R(t)$ is the cumulative distribution function of S_R. Also

$$Pr(\eta_t = R) = F_R(t) - F_{R+1}(t);\ F_0(t) = 1. \qquad \ldots(3.1)$$

Further, the total quantity demanded during time $t = NT$, is

$$\xi_t = \sum_{i=1}^{\eta_t} X_i \qquad \ldots(3.2)$$

the sum of a random number η_t of i.i.d. random variables $\{X_i\}$; it has a probability mass $1 - F_1(t)$ at $\xi = 0$ and for $\xi > 0$, it has a probability density

$$\sum_{n=1}^{\infty} g_n(\xi, N)\,[F_n(t) - F_{n+1}(t)] = \frac{d}{d\xi}\,\varphi\,[\xi(t)_{,}\,N], \text{ say} \qquad \ldots(3.3)$$

where $g_n(\xi, N)$ is the n-fold convolution* with itself of $g(\xi, N)$, the p.d.f of a single demand, i.e.

$$g_{n+1}(\xi, N) = \int_0^{\xi} g_n(\xi - u, N)\, g(u, N)\, du,\; g_1(\xi, N) = g(\xi, N). \qquad \ldots(3.4)$$

The expected revenue per period of fixed length T is

$$P(Q, N) = r \int_0^{\infty} \min(\xi, Q)\, d_{\xi}\varphi\,[\xi(T), N] - CQ - WQ - \frac{A}{N}\cdot \qquad \ldots(3.5)$$

Denote the minimum revenue function by $P_0(Q, N)$, i.e.

$$P_0(Q, N) = \min_{\varphi \in D} P(Q, N) \qquad \ldots(3.6)$$

where D is the class of distribution functions for which

$$\int_0^{\infty} \xi d_{\xi}\varphi\,[\xi(T), N] = \mu(N)$$

and

$$\int_0^{\infty} [\xi - \mu(N)]^2\, d_{\xi}\varphi\,[\xi(T), N] = \sigma^2(N).$$

To obtain the optimum ordering policy we have to maximize (3.6) with respect to Q and N. Since $\varphi\,[\xi(T), N]$ is the unknown distribution function of the random variable $\xi(T)$, we may not be in a position to apply differential calculus direct to obtain optimal values of Q and N. To determine the optimal values of Q and N, the maxmin criterion of an inventory problem (Arrow *et al.* 1958) will be utilized. The method may be described as 'quasi maxmin', since the distribution which will minimize the expected revenue per period puts all its mass on two points. We shall choose a stockage policy so as to maximize the minimum profit amongst all distributions with given mean $\mu(N)$ and variance $\sigma^2(N)$, i.e. to determine

$$\max_{Q, N}\ \min_{\varphi \in D} \left[r \int_0^{\infty} \min(\xi, Q)\, d_{\xi}\varphi\,[\xi(T), N] - CQ - WQ - \frac{A}{N} \right]$$

where inter-demand time has known distribution.

4. Solution of the Model

$$\text{Now}\quad P_0(Q, N) = \min_{\varphi \in D} \left[r \int_0^{\infty} \min(\xi, Q)\, d_{\xi}\varphi\,[\xi(T), N] - CQ - WQ - \frac{A}{N} \right] \qquad \ldots(4.1)$$

*If μ and σ^2 are the mean and variance of $g(\xi, N)$ then mean and variance of $g_n(\xi, N)$ are $\mu_n = n\mu$ and $\sigma_n^2 = n\sigma^2$.

can be determined if we can find the distribution function φ for which

$$\int_0^\infty \min(\xi, Q)\, d_\xi \varphi\, [\xi(T), N] \text{ is minimum.} \qquad \ldots(4.2)$$

This is possible by the following lemma due to Arrow *et al.* (1958).

Lemma — Let Q, $\mu(N)$ and $\sigma^2(N)$ be given; then there exists a quadratic function $Z(\xi) = \alpha + \beta\xi + \gamma\xi^2$, such that $Z(\xi) \leqslant \min(\xi, Q)$ for $\xi \geqslant 0$ with equality holding at only two points a and b. Moreover, there exists a two-point distribution confined to a and b with mean $\mu(N)$ and variance $\sigma^2(N)$. Denote the two-point distribution by $\psi(\xi, N)$, then

$$\int_0^\infty \min(\xi, Q)\, d_\xi \varphi\, [\xi(T), N] \geqslant \int_0^\infty Z(\xi)\, d_\xi \varphi\, [\xi(T), N]$$

$$= \alpha + \beta\mu(N) + \gamma\, [\mu^2(N) + \sigma^2(N)].$$

Actually $\quad Q - a = b - Q = [\sigma^2 + (Q - \mu)^2]^{1/2},$

$$\alpha = \frac{a^2}{2(a-b)},\ \beta = \frac{b}{b-a},\ \gamma = \frac{1}{2(a-b)},\ Pr(\xi = a) = \frac{\mu - b}{a - b}, \qquad \ldots(4.2a)$$

so that $a > 0$ and $b > 0$ iff $Q > (\sigma^2 + \mu^2)/2\mu$.

And since $Z(\xi)$ and $\min(\xi, Q)$ are equal at the points where $\psi(\xi, N)$ has all its mass, and

$$\int_0^\infty \min(\xi, Q)\, d_\xi\psi(\xi, N) = \int_0^\infty Z(\xi)\, d_\xi\psi(\xi, N)$$

$$= \alpha + \beta\mu(N) + \gamma\, [\mu^2(N) + \sigma^2(N)]$$

$$= \tfrac{1}{2}(\mu + a), \text{ on substituting from (4.2a).}$$

From (3.5) and (3.6)

$$P_0(Q, N) + (C + W)\, Q + A/N = \gamma \min_{\varphi \in D} \int_0^\infty \min(\xi, Q)\, d_\xi \varphi\, [\xi(T), N]$$

$$= \gamma \int_0^\infty \min(\xi, Q) \sum_{n=1}^\infty g_n(\xi, N)\, [F_n(T) - F_{n+1}(T)]\, d\xi \qquad \ldots(4.3)$$

the inter-demand time distribution being known.

(i) Assume that for $i = 1, 2, \ldots,$

$$Q \leqslant \frac{\mu_i^2(N) + \sigma_i^2(N)}{2\mu_i(N)} = \frac{i\mu^2(N) + \sigma^2(N)}{2\mu(N)}, \qquad \ldots(4.4)$$

since

$$\mu_i = i\mu \quad \text{and} \quad \sigma_i^2 = i\sigma^2.$$

Then

$$P_0(Q, N) = r \sum_{i=1}^{\infty} [F_i(T) - F_{i+1}(T)] \left[\frac{i\mu^2(N)\, Q}{i\mu^2(N) + \sigma^2(N)}\right] - (C+W)\, Q - \frac{A}{N}. \quad \ldots(4.4)$$

(ii) Assume $Q > \dfrac{\mu_i^2(N) + \sigma_i^2(N)}{2\mu_i(N)}$, then

$$P_0(Q, N) = \frac{r}{2} \sum_{i=1}^{\infty} [F_i(T) - F_{i+1}(T)]\, [i\mu(N) + Q - \{(Q - i)\, \mu(N))^2 + i\sigma^2(N)\}^{1/2}] - (C + W)\, Q - \frac{A}{N}. \quad \ldots(4.5)$$

To determine those values of Q and N which maximize the minimum expected profit $P_0(Q, N)$, it may be observed that this function is a concave function of Q (Agin 1966) and has continuous derivatives at all points. We shall find it convenient to consider two cases separately.

Case I — For $i = 1, 2, \ldots$

$$\frac{(C + W)}{r\, [F_i(T) - F_{i+1}(T)]} \left[1 + \frac{\sigma_i^2(N)}{\mu_i^2(N)}\right] < 1.$$

Since $P_0(Q, N)$ is positive for large Q, it may be noticed that the maximum will be attained for

$$Q > \frac{\mu_i^2(N) + \sigma_i^2(N)}{2\mu_i(N)} = \frac{i\mu^2(N) + \sigma^2(N)}{2\mu(N)}.$$

Now

$$P_0(Q, N) = \frac{r}{2} \sum_{i=1}^{\infty} [F_i(T) - F_{i+1}(T)]\, [i\mu(N) + Q - \{(Q - i\mu(N))^2 + i\sigma^2(N)\}^{1/2}] - (C + W)\, Q - \frac{A}{N}.$$

Equating $\partial P_0(Q, N)/\partial Q$ to zero

$$\frac{2(C + W)}{r} = \sum_{i=1}^{\infty} \{F_i(T) - F_{i+1}(T)\} \left[1 - \frac{Q - i\mu(N)}{[\{Q - i\mu(N)\}^2 + i\sigma^2(N)]^{1/2}}\right]$$

$$= \sum_{n=0}^{\infty} F_{n+1}(T) \left[\frac{Q - n\mu(N)}{\sqrt{[Q - n\mu(N)]^2 + n\sigma^2(N)}} - \frac{Q - (n+1)\,\mu(N)}{\sqrt{[Q - (n+1)\,\mu(N)]^2 + (n+1)\,\sigma^2(N)}}\right]. \quad \ldots(4.6)$$

This will determine the value of Q in terms of N.

Let that value be $Q_0(N)$

Case II — For $i = 1, 2, \ldots,$

$$\frac{(C + W)}{r\,[F_i(T) - F_{i+1}(T)]} \left[1 + \frac{\sigma_i^2(N)}{\mu_i^2(N)}\right] > 1.$$

$P_0(Q, N)$ is a concave function and therefore is always positive except for $Q = 0$. In this case the optimal policy is to buy none of the stock.

The whole system is summarized thus:

'The stock level which maximizes the minimum expected revenue for all demand distributions with mean $\mu(N)$ and variance $\sigma^2(N)$ is given by $Q = 0$ or $Q_0(N)$, according as

$$\frac{(C + W)}{r\,[F_i(T) - F_{i+1}(T)]} \left[1 + \frac{\sigma_i^2(N)}{\mu_i^2(N)}\right] \geqslant 1 \quad \text{or} \quad < 1, \; i = 1, 2, \ldots$$

where $Q_0(N)$ is already determined'.

We now substitute $Q_0(N)$ for Q into the expression (4.5) for $P_0(Q, N)$ and denote this revenue function by $R_0(N)$; then

$$R_0(N) = \frac{r}{2} \sum_{i=1}^{\infty} [F_i(T) - F_{i+1}(t)]\,[i\mu(N) + Q_0(N)] - [\{(Q_0(N) - \mu_i(N))^2 + \sigma_i^2(N)\}^{1/2}] - (C + W)\,Q_0(N) - \frac{A}{N} \quad \ldots(4.8)$$

$R_0(N)$ no longer depends upon Q. N being a discrete variable, the optimal value of N say N_0 can be obtained so as to satisfy the following inequalities :

$$R_0(N_0 + 1) - R_0(N_0) \leqslant 0$$

$$R_0(N_0 - 1) - R_0(N_0) \leqslant 0.$$

Thus almost optimal values of Q and N are obtained.

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